Distributionally robust chance-constrained multicommodity network flow problem in time-varying networks: by column-generation approach

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Abstract. The multicommodity network flow problem is a classical issue in network optimization, where the objective is to route multiple commodities through interconnected nodes and arcs to minimize the overall flow cost. However, in practical scenarios, parameters such as arc capacities, node demands, and travel costs may be uncertain, and this uncertainty can significantly affect the optimal solution. To address this, several studies have developed methods to incorporate uncertainty into multicommodity network flow models. This study focuses on the discrete dynamic multicommodity flow (DDMF) problem with intermediate node storage, which aims to minimize the cost of network flow over time. To achieve this, the path-flow formulation of DDMF is considered for the minimum cost network flow problem under parameter uncertainty. The study explores different perspectives, including robust optimization, chance-constrained (CC) optimization, and distributionally robust chance-constrained (DRCC) optimization. Certain models are formulated for each perspective. Furthermore, the performance of the DRCC, CC, proposed robust counterpart (RC), and stochastic optimization (SO) methods is compared. Computational results demonstrate that the DRCC and RC models offer efficient approaches that require significantly fewer CPU times compared to the CC and SO models for solving uncertain DDMF problems in large-scale networks.

Keywords: Uncertainty modelling; Robust optimization; Chance-constrained optimization; Distributionally robust; Column-generation approach.

1. Introduction

The multicommodity network flow (MCNF) problem is a highly significant issue in the field of network flow due to its broad application in various domains, including communication systems, urban traffic systems, railway systems, and logistics systems. In MCNF problems, multiple commodities need to be transferred from specific sources to designated sinks within a network, with each arc having a specific capacity. Many researchers have focused on MCNF problems and have proposed different linear and non-linear models to address these problems. Notable works include Ford and Fulkerson [19], Hu [30], Nagamochi and Ibaraki [57], Kennington [33], Assad [2], McBride [50], Ouorou et al. [58], Guo and Niedermeier [27], Mahey and de Souza [48], Lee [41], Fakhri and Ghatee [17], Letchford and Salazar-González [42], Karsten et al. [32], and Kabadurmus and Smith [31].

The MCNF problem with uncertain parameters has garnered the attention of several researchers. In summary, three types of approaches can be utilized to model inaccurate data in MCNF problems: fuzzy strategies, stochastic optimization (SO) methods, and robust optimization-based techniques. For instance, Ghatee and Hashemi [23] addressed the discrete dynamic multicommodity flow (DDMF) problem for minimizing the cost of network flow. They considered fuzzy numbers to represent the travel cost and travel demands. Kureichik and Evgeniya [39] proposed a method to determine the maximum two-commodity flow when faced with fuzzy parameters, such as arc capacities, with vitality degree. Mejri et al. [52] focused on the discrete cost multicommodity flow problem with demand uncertainty. They presented a two-stage stochastic programming approach along with a simulation-optimization approach. Khezri and Khodayifar [36] focused on MCNF problem in the presence of uncertain parameters and various types of costs associated with each arc in the network. They presented a multi-objective approach to solving this problem, where the coefficients of the capacity constraints are modelled as random variables with the normal distribution, and the dependence between them has been modelled using an Archimedean copula. They applied copula theory and fuzzy programming approach to convert the uncertain multi-objective problem into a certain single-objective problem and employed the piecewise tangent approximation and the piecewise linear approximation methods to solving their presented models.

Robust optimization (RO) is a powerful technique that has been extensively studied and applied in various fields, including engineering, finance, and operations research. The primary concept behind robust optimization is to account for uncertainty in the data and represent it as a set of possible scenarios, each representing a potential realization of the uncertain parameters. The objective is to find an optimal solution that performs well under all or most of these scenarios. For more detailed information, refer to the works of Ben-Tal and Nemirovski ([6], [7], and [8]) and Bertsimas and Sim [10]. Calafiore and El Ghaoui [13] explored distributionally robust chance-constrained linear programs and established explicit convex conditions to ensure compliance with probability constraints. Mudchanatongsuk et al. [55] proposed a robust formulation for the multicommodity network design problem, considering cost and demand uncertainty, and employed a column generation procedure for its solution. Altin et al. [1] considered polyhedral uncertainty in traffic demands for the network loading problem and presented a concise formulation of the problem. They also used an efficient branch and cut algorithm to solve it. Additional studies on this topic include the works of Peng and Jiang [59] and Silva et al. [63].

In various real-world applications such as road or air traffic control, production systems, and communication networks, the flow of entities can change over time. Ford and Fulkerson [20] introduced the maximum dynamic flow problem, demonstrating its relation to the minimum cost flow problem in a time-expanded network. Burkard et al. [12] investigated the quickest path problem and presented a strongly polynomial algorithm to solve it. Klinz and Woeginger [38] addressed the DDMF and devised a greedy algorithm for its solution. Sedeño-Noda and D. González-Barrera [62] focused on the quickest path problem and proposed a label setting algorithm that improved upon existing algorithms in the literature. Khodayifar [36] developed a model based on dynamic path flows for the DDMF and devised an algorithm based on the decomposition principle to solve it. Glockner and Nemhauser [24] considered the multicommodity dynamic network flow problem with capacity uncertainty and employed a Lagrangian decomposition method for its solution. Topaloglu and Powell [64] introduced an iterative dynamic programming-based methodology to solve the dynamic minimum cost integer multicommodity flow problem in the presence of stochastic data.

Lee and Dong [40] proposed dynamic location and allocation models and developed a two-stage stochastic programming model for multi-period reverse logistic network design under uncertainties. They suggested a heuristic algorithm to solve this problem. Bozhenyuk et al. [11] addressed the DDMF problem with fuzzy parameters and proposed a solution method. Mattia [49] considered the robust network loading problem with polyhedral uncertainty in demands and proposed a branch and cut algorithm. Lu et al. [47] investigated a fuzzy intercontinental multi-modal routing problem with time and capacity uncertainties and used a defuzzification-based approach to solve a fuzzy mixed integer linear programming (MILP) model.

Rahmaniani et al. [61] described a Benders decomposition algorithm with acceleration techniques to solve the multicommodity capacitated network design problem with demand uncertainty. Mohammadi et al. [54] proposed a model for designing a reliable hazardous material transportation network under uncertainties and integrated chance-constrained programming with a possibilistic programming approach. They provided a solution framework. Bozhenyuk and Gerasimer [11] studied the two-commodity maximum dynamic flow problem with fuzzy arc capacities and crisp transit times using a fuzzy temporal graph. For additional studies, refer to Lin and Jaillet [46], Li et al. [44], Li and Lai [43], Charikar et al. [15], Khodayifar et al. [37], Rahmani [60], Mishra and Prakash Singh [53], Calvete et al. [14], Yeh et al. [66], Khanjani-Shiraz et al. [34], and El Khadiri and Yeh [16] and Van Ackooij et al. [65]. Table 1 provides a summary of relevant papers on multicommodity network flow problems in the literature.

Table 1. Review of some multicommodity network flow problems.

Year	Reference	Model features				Type of mathematical programming to deal with the parameter uncertainty	Solution method	Type of algorithm	Type of problem
		Static	Dynamic	Certain	Uncertain				
1958	Ford and Fulkerson [19]	*		*		-	Simplex	Exact	Maximum multicommodity flow
2001	Glockner and Nemhauser [24]		*		*	Stochastic	Compath decomposition	Heuristic	DDMF
2002	Fleischer and Skutella [18]		*	*		-	Condensed time-expanded network	FPTAS	Quickest multicommodity flow
2005	Mudchanatongsuk et al. [55]	*			*	Robust	Column generation	Exact	Network design
2006	Topaloglu and Powell [64]		*		*	Stochastic	Dynamic programming method	Approximation	Integer DDMF
2007	Hall et al. [28]		*	*		-	Time-expanded network	Exact	DDMF
2007	Hall et al. [28]		*	*		-	Time-expanded network	Greedy	Quickest multicommodity flow
2009	Ghatee and Hashemi [23]	*			*	Fuzzy	k-shortest path algorithms	Exact	Minimum cost MCNF
2009	Lee and Dong [40]		*		*	Stochastic	Sampling method	Heuristic	Location and allocation
2010	Li et al. [45]	*		*		-	Ant colony optimization (ACO)	Meta-heuristic	Minimum cost MCNF and minimum congestion

2011	Altin et al. [1]	*			*	Robust	Branch and cut	Heuristic	Network loading problem
2012	Grob and Skutella [26]		*	*		-	Dynamic programming with Meggido's parametric search [47]	FPTAS	Maximum multicommodity flow
2015	Letchford and Salazar-González [42]	*		*		-	Two-index or set partitioning formulations	Exact	Capacitated vehicle routing problem
2015	Karsten et al. [32]		*	*		-	Column generation	Exact	Resource-constrained shortest path problem
2016	Mattia [49]		*		*	Robust	Branch and cut	Exact and heuristic	Network loading problem
2016	Lu et al. [47]		*		*	Fuzzy	Defuzzification approach	Heuristic	Intercontinental multi-modal routing problem
2017	Rahmaniani et al. [61]	*			*	Stochastic	Benders decomposition	Exact	Multicommodity capacitated network design
2017	Mohammadi et al. [54]		*		*	Chance-constrained programming	A meta-heuristic algorithm based on a lower bound approach	Meta-heuristic	Minimum total risk
2017	Fragkos et al. [21]		*	*		-	Decomposition methods and heuristic algorithms	Exact and heuristic	Network design
2017	Kureichik and Evgeniya [39]		*		*	Fuzzy	Augmenting path	Exact	Maximum multicommodity flow
2017	Bozhenyuk et al. [11]		*		*	Fuzzy	Time-expanded network	Exact	Maximum multicommodity flow
2018	Ghasemi et al. [22]		*		*	Stochastic	Particle swarm optimization and genetic algorithm	Meta-heuristic	DDMF and the minimum amount of the shortage of relief supplies
2018	Zhang et al. [67]		*	*		-	Storage time aggregated graph	Exact	Maximum multicommodity flow

2018	Grande et al. [25]		*	*		-	Column generation	Exact	DDMF
2019	Khodayifar [36]		*	*		-	Decomposition principle	Exact	DDMF
2020	Mejri et al. [52]	*			*	Stochastic	Cut-generation, column generation and, Monte-Carlo simulation	Approximation	Minimum cost MCNF and the expected penalties of unmet multicommodity demands
	Current study		*		*	Distributionally robust chance-constrained	Decomposition principle	Exact	DDMF

DDMF: Discrete dynamic multicommodity flow for minimum cost network flow; FPTAS: Fully polynomial time approximation scheme.

Contribution of this paper. Given that the decision analysis based on the uncertain data is inevitable in many real-world applications, therefore, this study focuses on the DDMF in the presence of uncertain parameters, such as the cost uncertainty, the demand uncertainty and, the arc capacity uncertainty. For this purpose, we develop the dynamic path formulation of the DDMF model proposed by Khodayifar [35] in the case of parameter uncertainty. To study the parameter uncertainty in the DDMF, for the first time, we consider the viewpoint which deals with the chance constraints in the DDMF problem under the distribution uncertainty and obtain the deterministic restrictions such that the probability constraints are guaranteed. All proposed models in this paper are LP problems that can be solved with the existing algorithms in the LP context. Also, in this paper, according to the special structure of the proposed models, an algorithm based on the column-generation approach is provided for solving the proposed models. Finally, we show the efficiency of the proposed approach and compare the objective values and the CPU times of the proposed DRCC and SO method by using a numerical example and a number of experimental tests. We show that the proposed DRCC model requires significantly less CPU time than the SO model to solve the uncertain DDMF problem for large-scale networks. Also, experimental results show that the proposed solution method performs faster than the LP solver CPLEX.

The remainder of this paper is organized as follows: Section 2 provides a review of the preliminaries and basic definitions outlined in Khodayifar [36]. In Section 3, we examine the DDMF problem in the presence of data uncertainty from a distributionally robust chance-constrained perspective. Section 4 introduces a column-generation method for solving the proposed models. In Section 5, we present the results through a numerical example and several experimental tests. Finally, Section 6 concludes the paper.

2. Preliminaries and basic definitions

Consider a directed network, denoted by $G = (N, A, K, c, u, \tau, \mathcal{T})$, where N is the set of $|N| = n$ nodes, A is the set of $|A| = m$ arcs and, K is the set of $|K| = h$ commodities that must be transmitted over the network. Every commodity $k \in K$ has only one source $s_k^+ \in N$ and one sink $s_k^- \in N$ and, R_k is the amount of supply or demand of commodity $k \in K$. Suppose that T is the time horizon and the time is considered in discrete steps, i.e., $\mathcal{T} = \{0, 1, \dots, \mathrm{T}\}$. Also, τ_{ij} is the travel time on arc (i, j), i.e., one unit of the flow of commodity

k which leaves node i at time t, that flow arrives at node j at time $t + \tau_{ij}$. Assume that $c_{ij}^k(t)$ and $c_i^k(t)$ are the per-unit flow cost of commodity k on arc (i, j) at time $t \in \mathcal{T}$ and the per-unit stored flow cost of commodity k at node i from time $t - 1$ to t, respectively. Moreover, $u_{ij}(t), u_i^k(t)$, and τ_{ij} are the upper bound on the amount of flow that can be transmitted on arc (i, j) at time $t \in \mathcal{T}$, the upper bound on the amount of flow that can be stored in node i from time $t - 1$ to t, and the travel time on arc (i, j) at time t, respectively.

There are two types of formulations for the DDMF problem: the arc-flow formulation and the path-cycle flow formulation. In Khodayifar [36], the arc-flow formulation of the DDMF problem with storage at intermediate nodes was proposed as follows:

$$\min \sum_{t=0}^{T} \left(\sum_{k \in K} \sum_{(i,j) \in A} c_{ij}^k(t) x_{ij}^k(t) + \sum_{k \in K} \sum_{i \in N} c_i^k(t) y_i^k(t) \right)$$

$$s.t. \qquad (2.1)$$

$$\sum_{k \in K} x_{ij}^k(t) \le u_{ij}(t), \qquad \forall (i, j) \in A, t \in \mathcal{T},$$

$$\sum_{\{j:\,(i,j) \in A\}} x_{ij}^k(t) - \sum_{\{j:\,(j,i) \in A\}} x_{ji}^k\big(t - \tau_{ji}\big) + y_i^k(t) - y_i^k(t-1) = 0, \qquad \forall k \in K, i \in N - \{s_k^+, s_k^-\}, t \in \mathcal{T},$$

$$\sum_{t=0}^{T} \left(\sum_{\{j:\,(i,j) \in A\}} x_{ij}^k(t) - \sum_{\{j:\,(j,i) \in A\}} x_{ji}^k\big(t - \tau_{ji}\big) \right) = d_i^k, \qquad \forall i \in N, k \in K,$$

$$0 \le x_{ij}^k(t), 0 \le y_i^k(t) \le u_i^k(t), \qquad \forall (i, j) \in A, i \in N, k \in K, t \in \mathcal{T},$$

where $d_i^k = \begin{cases} R_k & i = s_k^+ \\ -R_k & i = s_k^- \\ 0 & o.w. \end{cases}$

In model (2.1), the dynamic flow vector is represented by x^k, and the storage flow vector is represented by y^k, both in the time horizon T. The objective function of model (2.1) aims to minimize the total cost of the dynamic flow vector x_k and the storage flow vector y_k within the time horizon T. The first set of constraints imposes capacity limitations on the flow transferred on each arc $(i, j) \in A$ at each time step t. The second set of constraints enforces flow conservation and the amount of stored flow for each commodity $k \in K$ at each intermediate node i during each time step t. The third set of constraints ensures flow conservation throughout the entire time horizon T. The fourth set of constraints specifies the capacity limitations for $x_k (k \in K)$ and the storage capacity limitations for y_k $(k \in K)$.

According to the flow decomposition theorem, the arc-flow formulation of the DDMF problem (model 2.1) can be reformulated using path-cycle flows. The path-cycle flow formulation of the DDMF problem has a simpler constraint structure (block-diagonal) compared to the arc-flow formulation. Therefore, our focus is on the path-cycle flow formulation of the DDMF problem. In Khodayifar [36], several definitions are proposed, and an assumption is imposed to formulate the path-cycle flow formulation of the DDMF problem.

Assumption 1. For each commodity, the network does not contain a negative cycle, and this means that there is not a cycle with a negative length.

Assumption 1 suggests that in an optimal solution of the path-cycle flow formulation of the DDMF problem, the flow on each cycle is zero. As a result, one can eliminate the cycle flow variables in the path-cycle flow formulation by employing Assumption 1. Building on this assumption, Khodayifar [36] introduced the path-flow formulation of the DDMF problem.

Definition 1. (Khodayifar [36]) An $NTP\ (i,\alpha)$ is a node-time pair $(i,\alpha) \in N \times \mathcal{T}$ and shows a node in an arbitrary time step. For each $(i,j) \in A$, $NTP\ (i,\alpha)$ is arc-linked to the $NTP\ (j,\beta)$, if $\beta = \alpha + \tau_{ij}$. Also, the $NTP\ (i,\alpha)$ is node-linked to the $NTP\ (j,\beta)$ if $i = j$ and $\alpha \neq \beta$.

Definition 2. (Khodayifar [36]) For each commodity k, a dynamic path from node s_k^+ with departure time α to node s_k^- is a sequence of distinct NTPs (arc-linked or node-linked) as follows:

$$p^\alpha : (s_k^+, \alpha) = (i_1, t_1), (i_2, t_2), \dots, (i_r, t_r) = (s_k^-, \beta).$$

For each commodity k, P^k is the set of the dynamic paths from the source node s_k^+ to the sink node s_k^- in the network G on time horizon T.

For any dynamic path $p^\alpha \in P^k$ with the departure time α and for each $t \geq \alpha$ and all $(i,j) \in A$, the arc-path indicator variable, $\delta_{ij}(p^\alpha, t)$, is defined as follows:

$$\delta_{ij}(p^\alpha, t) = \begin{cases} 1 & (i,t) \text{ is arc} - \text{linked to } \left(j, t + \tau_{ij}\right) \text{ on dynamic path } p^\alpha \\ 0 & \text{o.w.} \end{cases}$$

and for each $t \geq \alpha$ and $i \in N$, the node-path indicator variable, $\gamma_i(p^\alpha, t)$, is defined as follows:

$$\gamma_i(p^\alpha, t) = \begin{cases} 1 & (i,t) \text{ is node} - \text{linked to } (i, t+1) \text{ on dynamic path } p^\alpha \\ 0 & \text{o.w.} \end{cases}$$

Hence, the cost of a dynamic path p^α for the commodity k, $c^k(p^\alpha)$, is defined as:

$$c^k(p^\alpha) = \sum_{t=0}^{T} \sum_{(i,j) \in A} \delta_{ij}(p^\alpha, t) c_{ij}^k(t) + \sum_{t=0}^{T} \sum_{i \in N} \gamma_i(p^\alpha, t) c_i^k(t) .$$

Khodayifar [36] proposed the dynamic path-flow formulation of DDMF as follows:

$$\begin{aligned}
&\xi_1^* = \min \sum_{k=1}^{h} \sum_{p \in P^k} \sum_{\alpha=0}^{T-\tau_p} c^k(p^\alpha) f(p^\alpha) \\
&s.t. \\
&\sum_{k=1}^{h} \sum_{p \in P^k} \sum_{\alpha=0}^{T-\tau_p} \delta_{ij}(p^\alpha, t) f(p^\alpha) \leq u_{ij}(t), && \forall (i,j) \in A, t \in \mathcal{T}, \\
&\sum_{p \in P^k} \sum_{\alpha=0}^{T-\tau_p} f(p^\alpha) = R_k, && \forall k \in K, \\
&\sum_{p \in P^k} \sum_{\alpha=0}^{T-\tau_p} \gamma_i(p^\alpha, t) f(p^\alpha) \leq u_i^k(t), && \forall i \in N - \{s_k^+, s_k^-\}, k \in K, t \in \mathcal{T}, \\
&f(p^\alpha) \geq 0, && \forall k \in K, p \in P^k, \alpha = 0,1, \dots, T - \tau_p,
\end{aligned} \tag{2.2}$$

where $\tau_p = \sum_{(i,j) \in p} \tau_{ij}$ is the total transit time of path p, and the decision variable $f(p^\alpha)$ is the flow on the dynamic path p with the departure time α from s_k^+.

Khodayifar [36] presented an algorithm based on the decomposition principle to solve model (2.2). The subsequent section focuses on developing the dynamic path formulation of the DDMF model (2.2) under parameter uncertainty with distributionally robust chance-constrained optimization technique.

3. Uncertain discrete dynamic multicommodity flow problem

In this section, we focus on model (2.2) and introduce uncertain parameters into this model while exploring distributionally robust chance-constrained optimization. Ben-Tal and Nemirovski (2000) emphasized that "if there is uncertainty in the data of an equality constraint, a good model-builder would not model the constraint as an equality, rather as a range constraint with the right-hand side bounds close to one another." Therefore, in our study of the dynamic multicommodity flow problem with uncertain parameters, we will make the following assumption within the network to formulate an equivalent model to model (2.2) in which all constraints are inequalities.

Assumption 2. The underlying network does not contain a negative dynamic path (i.e., a dynamic path with a negative length).

Theorem 1 formulates an equivalent model to model (2.2) by using Assumption 1 and Assumption 2.

Theorem 1. Model (2.2) is equivalent to the following model (3.1), if Assumption 1 and Assumption 2 hold.

$$\begin{aligned}
&\xi_2^* = \min \sum_{k=1}^{h} \sum_{p \in P^k} \sum_{\alpha=0}^{T-\tau_p} c^k(p^\alpha) f(p^\alpha) \\
&s.t. \\
&\sum_{k=1}^{h} \sum_{p \in P^k} \sum_{\alpha=0}^{T-\tau_p} \delta_{ij}(p^\alpha, t) f(p^\alpha) \le u_{ij}(t), && \forall (i,j) \in A, t \in \mathcal{T}, \\
&\sum_{p \in P^k} \sum_{\alpha=0}^{T-\tau_p} f(p^\alpha) \ge R_k, && \forall k \in K, \\
&\sum_{p \in P^k} \sum_{\alpha=0}^{T-\tau_p} \gamma_i(p^\alpha, t) f(p^\alpha) \le u_i^k(t), && \forall i \in N - \{s_k^+, s_k^-\}, k \in K, t \in \mathcal{T}, \\
&f(p^\alpha) \ge 0, && \forall k \in K, p \in P^k, \alpha = 0,1,\dots,T-\tau_p.
\end{aligned} \tag{3.1}$$

Proof. It is clear that $\xi_2^* \le \xi_1^*$. Conversely, we prove $\xi_1^* \le \xi_2^*$. We claim that, for each optimal solution of model (3.1), we have $\sum_{p \in P^k} \sum_{\alpha=0}^{T-\tau_p} f(p^\alpha) = R_k$. By contradiction, suppose that f^* is an optimal solution for model (3.1) such that

$$\exists k' \in K; \sum_{p \in P^{k'}} \sum_{\alpha=0}^{T-\tau_p} f^*(p^\alpha) > R_{k'},$$

where $f^*(p^\alpha)$ is the optimal flow on the dynamic path $p^\alpha \in P^{k'}$. Define $\vartheta = \sum_{p \in P^{k'}} \sum_{\alpha=0}^{T-\tau_p} f^*(p^\alpha) - R_{k'} > 0$. Suppose that there is $\tilde{p} \in P^{k'}$ with the departure time $\tilde{\alpha} \in \{0, \dots, T-\tau_p\}$, such that $\sum_{\alpha=0}^{T-\tau_p} f^*(\tilde{p}^\alpha) = \max_{p \in P^{k'}} \sum_{\alpha=0}^{T-\tau_p} f^*(p^\alpha)$ and $f^*(\tilde{p}^{\tilde{\alpha}}) = \max_{\{0,\dots,T-\tau_p\}} f^*(\tilde{p}^\alpha) = \beta$. We consider two of the following cases:

Case 1. $\beta \geq \vartheta$. In this case, we define a flow in the dynamic network as follows:

$$
\begin{aligned}
&f^{**}(p^{\alpha}) = f^{*}(p^{\alpha}), && \forall p \in P^{k}, \alpha \in \{0, \dots, T-\tau_p\}, && \forall k \neq k', \\
&f^{**}(p^{\alpha}) = f^{*}(p^{\alpha}), && \forall p \in P^{k'}, p \neq \tilde{p}, \alpha \in \{0, \dots, T-\tau_p\}, && \\
&f^{**}(\tilde{p}^{\alpha}) = f^{*}(\tilde{p}^{\alpha}), && \alpha \neq \tilde{\alpha}, && \\
&f^{**}(\tilde{p}^{\tilde{\alpha}}) = f^{*}(\tilde{p}^{\tilde{\alpha}}) - \vartheta \geq 0. && &&
\end{aligned}
$$

Therefore, f^{**} is a feasible solution for model (3.1) and $\sum_{k=1}^{h}\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} c^k(p^\alpha) f^{**}(p^\alpha) < \sum_{k=1}^{h}\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} c^k(p^\alpha) f^{*}(p^\alpha)$. By this contradiction, $\sum_{p\in P^{k'}}\sum_{\alpha=0}^{T-\tau_p} f^{*}(p^\alpha) = R_{k'}$.

Case 2. $\beta < \vartheta$, In this case, we define a flow in the network as follows:

$$
\begin{aligned}
&f^{**}(p^{\alpha}) = f^{*}(p^{\alpha}), && \forall p \in P^{k}, \alpha \in \{0, \dots, T-\tau_p\}, && \forall k \neq k', \\
&f^{**}(p^{\alpha}) = f^{*}(p^{\alpha}), && \forall p \in P^{k'}, p \neq \tilde{p}, \alpha \in \{0, \dots, T-\tau_p\}, && \\
&f^{**}(\tilde{p}^{\alpha}) = f^{*}(\tilde{p}^{\alpha}), && \alpha \neq \tilde{\alpha}, && \\
&f^{**}(\tilde{p}^{\tilde{\alpha}}) = f^{*}(\tilde{p}^{\tilde{\alpha}}) - \beta \geq 0. && &&
\end{aligned}
$$

Therefore, f^{**} is a feasible solution for model (3.1) and $\sum_{k=1}^{h}\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} c^k(p^\alpha) f^{**}(p^\alpha) < \sum_{k=1}^{h}\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} c^k(p^\alpha) f^{*}(p^\alpha)$. By this contradiction, in every optimal solution of model (3.1), we have:

$$\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} f(p^\alpha) = R_k, \quad \forall k \in K.$$

Therefore, $\xi_2^* \geq \xi_1^*$. Hence, model (2.2) is equivalent to model (3.1) in the presence of Assumption 1 and Assumption 2. □

In the following, we impose the uncertain parameters on model (3.1) and present the certain equivalent models for the DRCC problem from different perspectives.

3.1 Distributionally robust chance-constrained

In this section, we present explicit deterministic counterparts of the distributionally robust chance-constrained model for different families of probability distributions. Let $U_{ij}(t), U_k$, and $U_i^k(t)$ represent the families of probability distributions for $u_{ij}(t), R_k$, and $u_i^k(t)$, respectively, where $(i,j) \in A, i \in N, t \in T$, and $k \in K$. Accordingly, the DRCC model with confidence levels $\varepsilon, \varepsilon'$, and ε'' is as follows:

$$
\min \sum_{k=1}^{h} \sum_{p\in P^k} \sum_{\alpha=0}^{T-\tau_p} c^k(p^\alpha) f(p^\alpha)
$$

$s.t.$ (3.2)

$$
\inf_{u_{ij}(t)\sim U_{ij}(t)} \Pr\left(\sum_{k=1}^{h} \sum_{p\in P^k} \sum_{\alpha=0}^{T-\tau_p} \delta_{ij}(p^\alpha,t) f(p^\alpha) \leq u_{ij}(t) \right) \geq 1-\varepsilon_{ij}(t), \quad \forall (i,j)\in A, t\in \mathcal{T},
$$

$$
\inf_{R_k\sim U_k} \Pr\left(\sum_{p\in P^k} \sum_{\alpha=0}^{T-\tau_p} f(p^\alpha) \geq R_k \right) \geq 1-\varepsilon'_k, \quad \forall k\in K,
$$

$$
\inf_{u_i^k(t)\sim U_i^k(t)} \Pr\left(\sum_{p\in P^k} \sum_{\alpha=0}^{T-\tau_p} \gamma_i(p^\alpha,t) f(p^\alpha) \leq u_i^k(t) \right) \geq 1-\varepsilon''_{i,k}(t), \quad \forall i\in N-\{s_k^+, s_k^-\}, k\in K, t\in \mathcal{T},
$$

$$
f(p^\alpha) \geq 0, \quad \forall k\in K, p\in P^k, \alpha = 0,\dots,T-\tau_p.
$$

In the following, we discuss three classes of distributions. In Section 3.1.1, we examine the family of distributions on $u_{ij}(t), R_k$, and $u_i^k(t)$, for all $(i,j) \in A, i \in N, t \in T$, and $k \in K$, with given means and variances. Section 3.1.2 focuses on the family of distributions on $u_{ij}(t), R_k$, and $u_i^k(t)$, for all $(i,j) \in A, i \in N, t \in T, and\ k \in K$, defined over independent bounded intervals. In Section 3.1.3, we study a special family of distributions on $u_{ij}(t), R_k$, and $u_i^k(t)$, for all $(i,j) \in A, i \in N, t \in T$, and $k \in K$, known as radially symmetric non-increasing distributions.

3.1.1. DRCC in the presence of the family of distributions with known mean and variance (DRCC (type I))

In this section, we consider the family of distributions on $u_{ij}(t)$, which comprises all distributions with a given mean $\hat{u}_{ij}(t)$ and variance $\sigma^2_{u_{ij}(t)}$. Similarly, we consider the family of distributions on $u_i^k(t)$, which encompasses all distributions with a given mean $\hat{u}_i^k(t)$ and variance $\sigma^2_{u_i^k(t)}$. Additionally, we examine the family of distributions on R_k, which includes all distributions with a given mean $\hat{R}_k$ and variance $\sigma^2_{R_k(t)}$. The following theorem holds:

Theorem 2. The distributionally robust chance-constrained model (3.2), in the case of knowing the mean and variance, referred to as DRCC (type I), is equivalent to model (3.3):

$$\min \sum_{k=1}^{h} \sum_{p\in P^k} \sum_{\alpha=0}^{T-\tau_p} c^k(p^\alpha) f(p^\alpha)$$

$s.t.$ (3.3)

$$\sum_{k=1}^{h} \sum_{p\in P^k} \sum_{\alpha=0}^{T-\tau_p} \delta_{ij}(p^\alpha,t) f(p^\alpha) \geq \hat{u}_{ij}(t) - \sqrt{1-\varepsilon_{ij}(t)}\sigma^2_{u_{ij}(t)}, \quad \forall (i,j) \in A, t \in \mathcal{T},$$

$$\hat{R}_k + \sqrt{1-\varepsilon'_k}\sigma^2_{R_k} \geq \sum_{p\in P^k} \sum_{\alpha=0}^{T-\tau_p} f(p^\alpha), \quad \forall k \in K,$$

$$\sum_{p\in P^k} \sum_{\alpha=0}^{T-\tau_p} \gamma_i(p^\alpha,t) f(p^\alpha) \geq \hat{u}_i^k(t) - \sqrt{1-\varepsilon''_{i,k}(t)}\sigma^2_{u_i^k(t)}, \quad \forall i \in N - \{s_k^+, s_k^-\}, k \in K, t \in \mathcal{T},$$

$$f(p^\alpha) \geq 0, \quad \forall k \in K, p \in P^k, \alpha = 0, \dots, T-\tau_p.$$

Proof. Suppose that $\varepsilon_{ij}(t) \in (0,1)$. For all $(i,j) \in A, t \in \mathcal{T}$, we define, $-u_{ij}(t) = -\hat{u}_{ij}(t) + \sigma^2_{u_{ij}(t)} z_{ij}(t)$, where, $E\{z_{ij}(t)\} = 0, Var\{z_{ij}(t)\} = 1$. Now, we use the multivariate Chebyshev inequalities (Bertsimas and Popescu [9]) to obtain the following inequalities:

$$\sup_{u_{ij}(t)\sim U_{ij}(t)} \Pr\left(-\sum_{k=1}^{h} \sum_{p\in P^k} \sum_{\alpha=0}^{T-\tau_p} \delta_{ij}(p^\alpha,t) f(p^\alpha) < -\hat{u}_{ij}(t) + \sigma^2_{u_{ij}(t)} z_{ij}(t)\right) =$$

$$\sup_{u_{ij}(t)\sim U_{ij}(t)} \Pr\left(\hat{u}_{ij}(t) - \sum_{k=1}^{h} \sum_{p\in P^k} \sum_{\alpha=0}^{T-\tau_p} \delta_{ij}(p^\alpha,t) f(p^\alpha) < \sigma^2_{u_{ij}(t)} z_{ij}(t)\right) = \frac{1}{1+d^2_{ij}(t)},$$

where, $d^2_{ij}(t) = \inf_{\sigma^2_{u_{ij}(t)} z(t) > \hat{u}_{ij}(t) - A_{ij}(t)} |z(t)|^2$, such that, $A_{ij}(t) = \sum_{k=1}^{h} \sum_{p\in P^k} \sum_{\alpha=0}^{T-\tau_p} \delta_{ij}(p^\alpha,t) f(p^\alpha)$ for a fixed flow f. We consider two of the following cases:

a) $\hat{u}_{ij}(t) \leq A_{ij}(t)$. In this case, we can just take $z_{ij}(t) = 0$, and obtain the infimum $d^2_{ij}(t) = 0$.

b) $\hat{u}_{ij}(t) > A_{ij}(t)$. In this case, $\left(\frac{\hat{u}_{ij}(t) - A_{ij}(t)}{\sigma^2_{u_{ij}(t)}}\right)^2$ is the infimum for $|z_{ij}(t)|^2$. So,

$$d^2_{ij}(t) = \inf_{\sigma^2_{u_{ij}(t)} z(t) > \hat{u}_{ij}(t) - A_{ij}(t)} |z_{ij}(t)|^2 = \left(\frac{\hat{u}_{ij}(t) - A_{ij}(t)}{\sigma^2_{u_{ij}(t)}}\right)^2.$$

In summarizing, we have

$$d^2_{ij}(t) = \begin{cases} 0 & \hat{u}_{ij}(t) \leq A_{ij}(t) \\ \left(\frac{\hat{u}_{ij}(t) - A_{ij}(t)}{\sigma^2_{u_{ij}(t)}}\right)^2 & \hat{u}_{ij}(t) > A_{ij}(t) \end{cases}$$

Hence, the first constraint of the model (3.2) is satisfied if and only if $\hat{u}_{ij}(t) > \sum_{k=1}^{h} \sum_{p\in P^k} \sum_{\alpha=0}^{T-\tau_p} \delta_{ij}(p^\alpha,t) f(p^\alpha)$ and

$$\frac{1}{1+d_{ij}^2(t)} \leq \varepsilon_{ij}(t) \leftrightarrow \sqrt{1-\varepsilon_{ij}(t)}\sigma_{u_{ij}(t)}^2 \geq \hat{u}_{ij}(t) - \sum_{k=1}^{h}\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} \delta_{ij}(p^\alpha,t)f(p^\alpha).$$

By a similar argument as that above, the second constraint of the model (3.2) is satisfied if and only if $\hat{R}_k < \sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} f(p^\alpha)$ and $\sqrt{1-\varepsilon_k'}\sigma_{R_k}^2 \geq \sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} f(p^\alpha) - \hat{R}_k$.

Similarly, the third constraint of the model (3.2) is satisfied if and only if $\hat{u}_i^k(t) > \sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} \gamma_i(p^\alpha,t)f(p^\alpha)$, and

$$\sqrt{1-\varepsilon_{i,k}''(t)}\sigma_{u_i^k(t)}^2 \geq \hat{u}_i^k(t) - \sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} \gamma_i(p^\alpha,t)f(p^\alpha).$$

Hence, model (3.2) is equivalent to model (3.3), and this completes the proof. □

It is worth noting that, in practice, model (3.3) tends to be too large. It often contains thousands of columns, representing a seemingly unlimited number of dynamic paths. Consequently, solving the proposed model explicitly and utilizing existing algorithms in the LP context becomes a challenging task. To address this issue, we introduce a method based on the column-generation approach principle (inspired by the revised simplex method) in section 4.2. This method allows us to solve the model without explicitly enumerating all the dynamic paths.

3.1.2. DRCC in the case of the random data in independent intervals (DRCC (type II))

In this section, we analyze an uncertainty model where the random data $u_{ij}(t)$ have known mean $\hat{u}_{ij}(t)$ and the individual elements are only known to belong with probability one to independent bounded intervals; i.e., we assume that $u_{ij}(t) = \hat{u}_{ij}(t) + w_{ij}(t)$, where $w_{ij}(t) \in [l_{ij}^-(t), l_{ij}^+(t)]$ for all $(i,j) \in A, t \in \mathcal{T}$. Similarly, suppose that the random data R_k have known mean $\hat{R}_k$ and the individual elements are only known to belong with probability one to independent bounded intervals; i.e., we assume that $R_k = \hat{R}_k + s_k$, where $s_k \in [l_k^-, l_k^+]$ for all $k \in K$. Also, suppose that the random data $u_i^k(t)$ have known mean $\hat{u}_i^k(t)$ and the individual elements are only known to belong with probability one to independent bounded intervals; i.e., we assume that $u_i^k(t) = \hat{u}_i^k(t) + w_i^k(t)$, where $w_i^k(t) \in [l_i^{k-}(t), l_i^{k+}(t)]$ for all $i \in N, t \in \mathcal{T}, k \in K$. Let us denote $\overline{U}_{ij}(t), \overline{U}_k$ and $\overline{U}_i^k(t)$ are the families of the probability distributions on $u_{ij}(t), R_k$, and $u_i^k(t)$, for all $(i,j) \in A, i \in N, t \in \mathcal{T}$ and $k \in K$, respectively. By the above assumptions and notations, we express Theorem 3.

Theorem 3. For any confidence levels $\varepsilon_{ij}(t), \varepsilon_k', \varepsilon_{i,k}''(t) \in (0,1)$, the distributionally robust chance-constrained

$$\inf_{u_{ij}(t)\sim\overline{U}_{ij}(t)} \Pr\left(\sum_{k=1}^{h}\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} \delta_{ij}(p^\alpha,t)f(p^\alpha) \leq u_{ij}(t)\right) \geq 1-\varepsilon_{ij}(t),$$

holds if

$$\hat{u}_{ij}(t) + \sqrt{\frac{1}{2}\left(l_{ij}^+(t) - l_{ij}^-(t)\right)\ln\frac{1}{\left(1-\varepsilon_{ij}(t)\right)}} \geq \sum_{k=1}^{h}\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} \delta_{ij}(p^\alpha,t)f(p^\alpha).$$

And the distributionally robust chance-constrained

$$\inf_{R_k \sim \bar{U}_k} \Pr\left(\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} f(p^\alpha) \geq R_k\right) \geq 1-\varepsilon_k',$$

holds if

$$\sqrt{\frac{1}{2}(l_k^+ - l_k^-)\ln\left(\frac{1}{\varepsilon_k'}\right)} + \hat{R}_k \leq \sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} f(p^\alpha).$$

And the distributionally robust chance-constrained

$$\inf_{u_i^k(t)\sim \bar{U}_i^k(t)} \Pr\left(\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} \gamma_i(p^\alpha,t) f(p^\alpha) \leq u_i^k(t)\right) \geq 1-\varepsilon_{i,k}''(t),$$

holds if

$$\hat{u}_i^k(t) + \sqrt{\frac{1}{2}\left(l_i^{k+}(t) - l_i^{k-}(t)\right)\ln\frac{1}{\left(1-\varepsilon_{i,k}''(t)\right)}} \geq \sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} \gamma_i(p^\alpha,t) f(p^\alpha).$$

Proof. For all $(i,j)\in A, t\in \mathcal{T}$, we define, $u_{ij}(t) = \hat{u}_{ij}(t) + w_{ij}(t)$. Therefore, by applying the Hoeffding tail probability inequality (see Hoeffding [28]), we have:

$$1-\varepsilon_{ij}(t) \leq \Pr\left(\sum_{k=1}^{h}\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} \delta_{ij}(p^\alpha,t) f(p^\alpha) - \hat{u}_{ij}(t) \leq w_{ij}(t)\right) \leq e^{\frac{-2\left(\sum_{k=1}^{h}\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p}\delta_{ij}(p^\alpha,t)f(p^\alpha)-\hat{u}_{ij}(t)\right)^2}{(l_{ij}^+(t)-l_{ij}^-(t))}}.$$

It is clear that, the following equivalence relation is held:

$$e^{\frac{-2\left(\sum_{k=1}^{h}\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p}\delta_{ij}(p^\alpha,t)f(p^\alpha)-\hat{u}_{ij}(t)\right)^2}{(l_{ij}^+(t)-l_{ij}^-(t))}} \geq 1-\varepsilon_{ij}(t) \leftrightarrow$$

$$\hat{u}_{ij}(t) + \sqrt{\frac{1}{2}\left(l_{ij}^+(t) - l_{ij}^-(t)\right)\ln\frac{1}{\left(1-\varepsilon_{ij}(t)\right)}} \geq \sum_{k=1}^{h}\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} \delta_{ij}(p^\alpha,t) f(p^\alpha).$$

This means that, if $\hat{u}_{ij}(t) + \sqrt{\frac{1}{2}\left(l_{ij}^+(t) - l_{ij}^-(t)\right)\ln\frac{1}{\left(1-\varepsilon_{ij}(t)\right)}} \geq \sum_{k=1}^{h}\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} \delta_{ij}(p^\alpha,t) f(p^\alpha)$, then

$$\inf_{u_{ij}(t)\sim \bar{U}_{ij}(t)} \Pr\left(\sum_{k=1}^{h}\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} \delta_{ij}(p^\alpha,t) f(p^\alpha) \leq u_{ij}(t)\right) \geq 1-\varepsilon_{ij}(t).$$

By a similar argument as that above, if $\sqrt{\frac{1}{2}(l_k^+ - l_k^-)\ln\left(\frac{1}{\varepsilon_k'}\right)} + \hat{R}_k \leq \sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} f(p^\alpha)$, then, we have $\inf_{R_k\sim \bar{U}_k} \Pr\left(\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} f(p^\alpha) \geq R_k\right) \geq 1-\varepsilon_k'$.

Similarly, for all $i \in N, t \in \mathcal{T}$, if $\hat{u}_i^k(t) + \sqrt{\frac{1}{2}\left(l_i^{k+}(t) - l_i^{k-}(t)\right)\ln\frac{1}{\left(1-\varepsilon_{i,k}''(t)\right)}} \geq \sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p}\gamma_i(p^\alpha,t)f(p^\alpha)$, then, we have:

$$\inf_{u_i^k(t)\sim\bar{U}_i^k(t)} \Pr\left(\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p}\gamma_i(p^\alpha,t)f(p^\alpha) \leq u_i^k(t)\right) \geq 1-\varepsilon_{i,k}''(t).$$

It completes the proof. This means that, in the presence of the random parameter in the independent intervals, the distributionally robust chance-constrained model, in the case of independent intervals, denoted by DRCC (type II), is as follows:

$$\begin{aligned}
&\min \sum_{k=1}^{h}\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} c^k(p^\alpha)f(p^\alpha) && \\
&s.t. && (3.4)\\
&\hat{u}_{ij}(t) + \sqrt{\frac{1}{2}\left(l_{ij}^{+}(t) - l_{ij}^{-}(t)\right)\ln\frac{1}{\left(1-\varepsilon_{ij}(t)\right)}} \geq \sum_{k=1}^{h}\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p}\delta_{ij}(p^\alpha,t)f(p^\alpha), && \forall (i,j)\in A, t\in\mathcal{T},\\
&\sqrt{\frac{1}{2}(l_k^{+} - l_k^{-})\ln\left(\frac{1}{\varepsilon_k'}\right)} + \hat{R}_k \leq \sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} f(p^\alpha), && \forall k\in K,\\
&\hat{u}_i^k(t) + \sqrt{\frac{1}{2}\left(l_i^{k+}(t) - l_i^{k-}(t)\right)\ln\frac{1}{\left(1-\varepsilon_{i,k}''(t)\right)}} \geq \sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p}\gamma_i(p^\alpha,t)f(p^\alpha), && \forall i\in N-\{s_k^+,s_k^-\}, k\in K, t\in\mathcal{T},\\
&f(p^\alpha)\geq 0, && \forall k\in K, p\in P^k, \alpha=0,\dots,T-\tau_p,
\end{aligned}$$

which is an LP problem.

3.1.3. DRCC in the case of the radially symmetric non-increasing distributions (DRCC (type III))

In this section, we analyze an uncertainty model where the random data $u_{ij}(t)$ have known mean $\hat{u}_{ij}(t)$, and the individual elements are only known to belong to $E_{u_{ij}(t)} = \{u_{ij}(t) | u_{ij}(t) = \hat{u}_{ij}(t) + q_{ij}(t)w_{ij}(t), |w_{ij}(t)| \leq 1\}$ for all $(i,j)\in A, t\in\mathcal{T}$, where $q_{ij}(t)$ is a scalar and $w_{ij}(t)$ is a random variable with the probability density as follows:

$$f_{ij,t}\left(w_{ij}(t)\right) = \begin{cases} g_{ij,t}^1(|w_{ij}(t)|) & |w_{ij}(t)|\leq 1\\ 0 & o.w.\end{cases}$$

Similarly, suppose that the random data R_k have known mean $\hat{R}_k$ and the individual elements are only known to belong to $E_{R_k} = \{R_k | R_k = \hat{R}_k + p_k s_k, |s_k|\leq 1\}$ for all $k\in K$, where p_k is a scalar and s_k is a random variable with the probability density as follows:

$$f_k(s_k) = \begin{cases} h(|s_k|) & |s_k|\leq 1,\\ 0 & o.w.\end{cases}$$

Also, suppose that the random data $u_i^k(t)$ have known mean $\hat{u}_i^k(t)$ and the individual elements are only known to belong to $E_{u_i^k(t)} = \{u_i^k(t) | u_i^k(t) = \hat{u}_i^k(t) + q_i^k(t)w_i^k(t), |w_i^k(t)|\leq 1\}$ for all $i\in N, t\in\mathcal{T}, k\in K$, where $q_i^k(t)$ is a scalar and $w_i^k(t)$ is a random variable with the probability density as follows:

$$f_{i,k,t}\left(w_i^k(t)\right) = \begin{cases} g_{i,k,t}^2\left(\left|w_i^k(t)\right|\right) & \left|w_i^k(t)\right| \le 1, \\ 0 & o.w. \end{cases}$$

where $g_{ij,t}^1(\cdot), g_{i,k,t}^2(\cdot)$, and $h_k(\cdot)$ are nonincreasing functions. Let us denote $U_{E_{ij}(t)}, U_{E_{R_k}}$, and $U_{E_i^k(t)}$ are the families of the probability distributions on $u_{ij}(t), R_k$, and $u_i^k(t)$, for all $(i,j) \in A, i \in N, t \in \mathcal{T}$, and $k \in K$, respectively.

Theorem 4. Suppose that $u_{ij}(t), R_k$, and $u_i^k(t)$ are random variables in the ellipsoids, $E_{u_{ij}(t)}, E_{R_k}$, and $E_{u_i^k(t)}$, respectively. For any $\varepsilon_{ij}(t), \varepsilon_k', \varepsilon_{i,k}''(t) \in (0, 0.5]$, the uniform distribution is the worst-case distribution. This means that

$$\inf_{u_{ij}(t)\sim U_{E_{ij}(t)}} \Pr\left(\sum_{k=1}^{h}\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} \delta_{ij}(p^\alpha, t) f(p^\alpha) \le u_{ij}(t)\right) \ge 1 - \varepsilon_{ij}(t),$$

is equivalent to the chance-constrained

$$\Pr\left(\sum_{k=1}^{h}\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} \delta_{ij}(p^\alpha, t) f(p^\alpha) \le u_{ij}(t)\right) \ge 1 - \varepsilon_{ij}(t),$$

where $u_{ij}(t)$ has the uniform distribution. Also,

$$\inf_{R_k\sim U_{E_{R_k}}} \Pr\left(\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} f(p^\alpha) \ge R_k\right) \ge 1 - \varepsilon_k',$$

is equivalent to the chance constraint

$$\Pr\left(\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} f(p^\alpha) \ge R_k\right) \ge 1 - \varepsilon_k',$$

where R_k has the uniform distribution and

$$\inf_{u_i^k(t)\sim U_{E_i^k(t)}} \Pr\left(\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} \gamma_i(p^\alpha, t) f(p^\alpha) \le u_i^k(t)\right) \ge 1 - \varepsilon_{i,k}''(t),$$

is equivalent to the chance constraint $\Pr\left(\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} \gamma_i(p^\alpha, t) f(p^\alpha) \le u_i^k(t)\right) \ge 1 - \varepsilon_{i,k}''(t)$, where $u_i^k(t)$ has the uniform distribution.

Proof. Regarding Theorem 6.3 in Barmish et al. [4], it suffices to show that the following sets are star-shaped.

$$\Omega_{ij,t}^1 = \left\{ w_{ij}(t) \colon \hat{u}_{ij}(t) + q_{ij}(t) w_{ij}(t) \ge \sum_{k=1}^{h}\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} \delta_{ij}(p^\alpha, t) f(p^\alpha) \right\},$$

$$\Omega_{i,k,t}^2 = \left\{ w_i^k(t) \colon \hat{u}_i^k(t) + q_i^k(t) w_i^k(t) \ge \sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} \gamma_i(p^\alpha, t) f(p^\alpha) \right\},$$

$$\Omega_k^3 = \left\{ s_k \colon \hat{R}_k + p_k s_k \le \sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} f(p^\alpha) \right\}.$$

Given that $u_{ij}(t)$ has a symmetric distribution, therefore, we can obtain:

$$\Pr\left(u_{ij}(t) \geq \sum_{k=1}^{h}\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p}\delta_{ij}(p^\alpha,t)f(p^\alpha)\right) \geq 0.5 \leftrightarrow \hat{u}_{ij}(t) \geq \sum_{k=1}^{h}\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p}\delta_{ij}(p^\alpha,t)f(p^\alpha).$$

Therefore, $\varepsilon_{ij}(t) \in (0, 0.5)$ requires that $\hat{u}_{ij}(t) \geq \sum_{k=1}^{h}\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p}\delta_{ij}(p^\alpha,t)f(p^\alpha)$. Hence, $0 \in \Omega_{ij,t}^1$, if $w \in \Omega_{ij,t}^1$ and $\rho \in [0,1]$, then, we have

$$\hat{u}_{ij}(t) + q_{ij}(t)w_{ij}(t) \geq \sum_{k=1}^{h}\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p}\delta_{ij}(p^\alpha,t)f(p^\alpha) \rightarrow \rho q_{ij}(t)w_{ij}(t) \geq \rho\left(\sum_{k=1}^{h}\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p}\delta_{ij}(p^\alpha,t)f(p^\alpha) - \hat{u}_{ij}(t)\right)$$

$$\geq \sum_{k=1}^{h}\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p}\delta_{ij}(p^\alpha,t)f(p^\alpha) - \hat{u}_{ij}(t).$$

Therefore, $\Omega_{ij,t}^1$ is star-shaped. By a similar argument as that above, $\Omega_{i,k,t}^2$ and Ω_k^3 are star-shaped, and the proof is completed. □

Theorem 4 demonstrates the equivalence between the DRCC model, considering the family of probability distributions discussed in this section, and the chance-constrained model under uniform distributions. The following theorem presents a certain equivalent model for the chance-constrained model when the uniform distributions are considered.

Theorem 5. Suppose that $u_{ij}(t) - \hat{u}_{ij}(t), u_i^k(t) - \hat{u}_i^k(t)$, and $R_k - \hat{R}_k$, for all $(i,j) \in A, t \in \mathcal{T}$, and $k \in K$, be uniformly distributed in the ellipsoids, $E_{u_{ij}(t)}, E_{u_i^k(t)}$, and E_{R_k}, respectively. Then, The DRCC (type III) model is equivalent to:

$$\begin{aligned}
&\min \sum_{k=1}^{h}\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} c^k(p^\alpha)f(p^\alpha) \\
&s.t. && (3.5)\\
&\sum_{k=1}^{h}\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p}\delta_{ij}(p^\alpha,t)f(p^\alpha) \leq u_{ij}^3(t), && \forall (i,j) \in A, t \in \mathcal{T},\\
&\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} f(p^\alpha) \geq R_k^3, && \forall k \in K,\\
&\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p}\gamma_i(p^\alpha,t)f(p^\alpha) \leq u_i^{3k}(t), && \forall i \in N - \{s_k^+, s_k^-\}, k \in K, t \in \mathcal{T},\\
&f(p^\alpha) \geq 0, && \forall k \in K, p \in P^k, \alpha = 0,1,\dots,T-\tau_p.
\end{aligned}$$

where $u_{ij}^3(t) = \hat{u}_{ij}(t) - q_{ij}(t)\left(1 - 2\varepsilon_{ij}(t)\right), R_k^3 = p_k(1 - 2\varepsilon_k') + \hat{R}_k$ and $u_i^{3k}(t) = u_i^k(t) - q_i^k(t)\left(1 - 2\varepsilon_{i,k}''(t)\right)$.

Proof. Suppose that $u_{ij}(t) - \hat{u}_{ij}(t), u_i^k(t) - \hat{u}_i^k(t)$, and $R_k - \hat{R}_k$, for all $(i,j) \in A, t \in \mathcal{T}$, and $k \in K$, be uniformly distributed in the following ellipsoids:

$$E_{u_{ij}(t)} = \{u_{ij}(t) | u_{ij}(t) = \hat{u}_{ij}(t) + q_{ij}(t) w_{ij}(t), |w_{ij}(t)| \leq 1\},$$
$$E_{u_i^k(t)} = \{u_i^k(t) | u_i^k(t) = \hat{u}_i^k(t) + q_i^k(t) w_i^k(t), |w_i^k(t)| \leq 1\},$$
$$E_{R_k} = \{R_k | R_k = \hat{R}_k + p_k s_k, |s_k| \leq 1\},$$

where $w_{ij}(t), w_i^k(t)$, and s_k are the random variables with the probability density as follows:

$$f_{ij,t}(w_{ij}(t)) = \begin{cases} g_{ij,t}^1(|w_{ij}(t)|) & |w_{ij}(t)| \leq 1, \\ 0 & o.w. \end{cases}$$

$$f_{i,k,t}(w_i^k(t)) = \begin{cases} g_{i,k,t}^2(|w_i^k(t)|) & |w_i^k(t)| \leq 1, \\ 0 & o.w. \end{cases}$$

$$f_k(s_k) = \begin{cases} h(|s_k|) & |s_k| \leq 1, \\ 0 & o.w. \end{cases}$$

Without loss of generality, suppose that $q_{ij}(t) \geq 0$, therefore, we have:

$$\hat{u}_{ij}(t) - q_{ij}(t) \leq u_{ij}(t) \leq \hat{u}_{ij}(t) + q_{ij}(t) \leftrightarrow -\hat{u}_{ij}(t) - q_{ij}(t) \leq -u_{ij}(t) \leq -\hat{u}_{ij}(t) + q_{ij}(t).$$

Given that $u_{ij}(t)$ is uniformly distributed, therefore $-u_{ij}(t)$ is uniformly distributed in $[-\hat{u}_{ij}(t) - q_{ij}(t), -\hat{u}_{ij}(t) + q_{ij}(t)]$, hence

$$F_{-u_{ij}(t)}^{-1}(1 - \varepsilon_{ij}(t)) = \left\{-u_{ij}(t) : \frac{-u_{ij}(t) + \hat{u}_{ij}(t) + q_{ij}(t)}{2q_{ij}(t)} = 1 - \varepsilon_{ij}(t)\right\} \rightarrow$$
$$-u_{ij}(t) = 2q_{ij}(t)(1 - \varepsilon_{ij}(t)) - \hat{u}_{ij}(t) - q_{ij}(t) = q_{ij}(t)(1 - 2\varepsilon_{ij}(t)) - \hat{u}_{ij}(t) \rightarrow$$
$$F_{-u_{ij}(t)}^{-1}(1 - \varepsilon_{ij}(t)) = q_{ij}(t)\left(1 - 2\varepsilon_{ij}(t)\right) - \hat{u}_{ij}(t).$$

Therefore,

$$-\sum_{k=1}^{h} \sum_{p \in P^k} \sum_{\alpha=0}^{T-\tau_p} \delta_{ij}(p^\alpha, t) f(p^\alpha) \geq F_{-u_{ij}(t)}^{-1}(1 - \varepsilon_{ij}(t)) \leftrightarrow \sum_{k=1}^{h} \sum_{p \in P^k} \sum_{\alpha=0}^{T-\tau_p} \delta_{ij}(p^\alpha, t) f(p^\alpha) \leq \hat{u}_{ij}(t) - q_{ij}(t)\left(1 - 2\varepsilon_{ij}(t)\right).$$

By a similar argument as that above, the following equivalence relations are held:

$$\sum_{p \in P^k} \sum_{\alpha=0}^{T-\tau_p} f(p^\alpha) \geq F_{R_k}^{-1}(1 - \varepsilon_k') \leftrightarrow \sum_{p \in P^k} \sum_{\alpha=0}^{T-t_p} f(p^\alpha) \geq p_k(1 - 2\varepsilon_k') + \hat{R}_k,$$

and

$$-\sum_{p \in P^k} \sum_{\alpha=0}^{T-\tau_p} \gamma_i(p^\alpha, t) f(p^\alpha) \geq F_{-u_i^k(t)}^{-1}(1 - \varepsilon_{i,k}'') \leftrightarrow \sum_{p \in P^k} \sum_{\alpha=0}^{T-\tau_p} \gamma_i(p^\alpha, t) f(p^\alpha) \leq u_i^k(t) - q_i^k(t)(1 - 2\varepsilon_{i,k}'').$$

Therefore, the chance-constrained problem under the uniform distributions on the random variables is as follows:

$$\min \sum_{k=1}^{h} \sum_{p\in P^k} \sum_{\alpha=0}^{T-\tau_p} c^k(p^\alpha) f(p^\alpha) \qquad (3.6)$$

$s.t.$

$$\sum_{k=1}^{h} \sum_{p\in P^k} \sum_{\alpha=0}^{T-\tau_p} \delta_{ij}(p^\alpha,t) f(p^\alpha) \le \hat{u}_{ij}(t) - q_{ij}(t)\left(1-2\varepsilon_{ij}(t)\right), \quad \forall (i,j)\in A, t\in\mathcal{T},$$

$$\sum_{p\in P^k} \sum_{\alpha=0}^{T-\tau_p} f(p^\alpha) \ge p_k(1-2\varepsilon'_k) + \hat{R}_k, \quad \forall k\in K,$$

$$\sum_{p\in P^k} \sum_{\alpha=0}^{T-\tau_p} \gamma_i(p^\alpha,t) f(p^\alpha) \le u_i^k(t) - q_i^k(t)(1-2\varepsilon''_{i,k}), \quad \forall i\in N-\{s_k^+,s_k^-\}, k\in K, t\in\mathcal{T},$$

$$f(p^\alpha)\ge 0, \quad \forall k\in K, p\in P^k, \alpha=0,1,\dots,T-\tau_p.$$

4. Solution method

As mentioned earlier, the proposed models in this paper are often too large to be explicitly solved. Therefore, finding an optimal solution without enumerating all the dynamic paths poses a significant challenge. To address this, we have developed a method based on the column-generation approach, inspired by the revised simplex method, as described in Section 4.2. Our focus is on the desired model (3.6).

In each step of the revised simplex method, a basis is determined and a vector of dual variables is determined by this basis for the constraints of the primal model. Suppose that the dual variables corresponding to the constraints $1^{th}, 2^{th}$, and 3^{th} of the model (3.6) are $w_{ij}(t), \sigma^k$, and $v_i^k(t)$, respectively. Regarding the dual variables, the reduced cost, denoted by $C_{p^\alpha}^{w,\sigma,v}$, for the k^{th} commodity and the dynamic path p^α,

$$p^\alpha: (s_k^+,\alpha) = (i_1,t_1),(i_2,t_2),\dots,(i_r,t_r) = (s_k^-,\beta),$$

from $NTP\ (s_k^+,\alpha)$ to $NTP\ (s_k^-,\beta)$ is as follows:

$$C_{p^\alpha}^{w,\sigma,v} = \sum_{(i_r,i_{r+1})\in P^\alpha, i_r\neq i_{r+1}} \left(w_{i_r,i_{r+1}}(t_r) + c^k_{i_r,i_{r+1}}(t_r)\right) + \sum_{(i_r,i_{r+1})\in P^\alpha, i_r = i_{r+1}} \left({v^k}_{i_r}(t_r) + c^k_{i_r}(t_r)\right) - \sigma^k.$$

It should be noted that, for a dynamic path $p\in P^k$, with the departure time α, the reduced cost $C_{p^\alpha}^{w,\sigma,v}$ is the cost of the dynamic path with the modified costs $w_{i_r,i_{r+1}}(t_r) + c^k_{i_r,i_{r+1}}(t_r) + v_{i_r}(t_r) + c^k_{i_r}(t_r)$ minus the commodity cost σ^k.

According to the complementary slackness conditions, the dynamic path flows $f(p^\alpha)$ in model (3.6) are optimal if and only if the following conditions are held:

(1) $w_{ij}(t)\left(\sum_{k=1}^{h}\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p}\delta_{ij}(p^\alpha,t)f(p^\alpha) - \hat{u}_{ij}(t) + q_{ij}(t)\left(1-2\varepsilon_{ij}(t)\right)\right) = 0, \quad \forall (i,j)\in A, t\in\mathcal{T},$

(2) $v_i^k(t)\left(\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p}\gamma_i(p^\alpha,t)f(p^\alpha) - u_i^k(t) + q_i^k(t)(1-2\varepsilon''_{i,k})\right) = 0, \quad \forall k\in K, i\in N-\{s_k^+,s_k^-\}, t\in\mathcal{T},$

(3) $\left(\sum_{p\in P^k}\sum_{\alpha=0}^{T-\tau_p} f(p^\alpha) - p_k(1-2\varepsilon'_k) - \hat{R}_k\right)\sigma^k = 0,$

(4) $C_{p^\alpha}^{w,\sigma,v}\ge 0, \quad \forall k\in K, p\in P^k, \alpha=0,1,\dots,T-\tau_p,$

(5) $f(p^\alpha)C_{p^\alpha}^{w,\sigma,v} = 0, \quad \forall k\in K, p\in P^k, \alpha=0,1,\dots,T-\tau_p.$

The fourth condition corresponds to

$\sum_{(i_r,i_{r+1})\in P^\alpha, i_r\neq i_{r+1}} \left(w_{i_r,i_{r+1}}(t_r) + c^k_{i_r,i_{r+1}}(t_r)\right) + \sum_{(i_r,i_{r+1})\in P^\alpha, i_r = i_{r+1}} \left(v_{i_r}(t_r) + c^k_{i_r}(t_r)\right) \geq \sigma^k$. The fifth condition shows that the reduced cost $C^{w,\sigma,v}_{p^\alpha}$ should be zero for each dynamic path p^α with $f(p^\alpha) > 0$. So, the following Theorem is obtained by conditions (4) and (5).

Theorem 6. At the optimality of model (3.6), for the k^{th} commodity, the dynamic path p^α

$$p^\alpha : (s_k^+, \alpha) = (i_1, t_1), (i_2, t_2), \dots, (i_r, t_r) = (s_k^-, \beta),$$

from $NTP\ (s_k^+, \alpha)$ to $NTP\ (s_k^-, \beta)$ with $f(p^\alpha) > 0$ is a dynamic shortest path concerning the modified costs $w_{i_r,i_{r+1}}(t_r) + c^k_{i_r,i_{r+1}}(t_r) + v_{i_r}(t_r) + c^k_{i_r}(t_r)$. Also, the length of this dynamic shortest path is equal to commodity cost σ^k.

Proof. This is straightforward from the conditions (1)-(5).

4.1. Column generation procedure

This section proposes a method based on the column-generation approach for solving model (3.6) using Theorem 6 and the revised simplex method.

As we have noted, in each step of the revised simplex method, a basis is determined. Given that the reduced cost of each variable on this basis is zero, we can compute the arc prices $w_{ij}(t)$, commodity prices σ^k, and node-commodity prices $v_i^k(t)$. In other words, if the dynamic path p^α from $NTP\ (s_k^+, \alpha)$ to $NTP\ (s_k^-, \beta)$ for commodity k is a basic variable, then $C^{w,\sigma,v}_{p^\alpha} = 0$.

As is well known, in the revised simplex method, some basis always satisfies the conditions (1), (2), (3), and (5). Hence, a basis is optimal if the fourth condition is satisfied. In other words , it is enough to check the dual feasibility condition. In the following, the pseudocode of the column generation approach to solve (3.6) is summarized in Algorithm 1.

5. Computational Results

This section presents the computational results to demonstrate the proposed distributionally robust chance-constrained models to solve the DDMF problem in the presence of uncertain parameters. To test the performance of the proposed methods, we consider two case studies: (i) a general illustrative network and (ii) the different classes of randomly generated instances. All the test instances are carried out on a core i5-4670 and 3.40 GHz computer with 8.00 GB RAM. The proposed algorithm and models are coded in GAMS 24.1, and CPLEX is used as the optimization solver for solving subproblems in Algorithm 1, respectively. The default setting is used in our runs. It is worth noting that the reported CPU times for all examples are based on Algorithm 1, except for the CPU times of the fifth column in Table 4.

Algorithm 1: Column generation approach

Step 0: (Initialization)

Start with a basic feasible solution and find the arc prices $w_{ij}(t)$, commodity prices σ^k, and node-commodity prices $v_i^k(t)$ by the equations

$\sum_{(i_r,i_{r+1})\in P^\alpha, i_r\neq i_{r+1}} \left(w_{i_r,i_{r+1}}(t_r) + c^k_{i_r,i_{r+1}}(t_r)\right) + \sum_{(i_r,i_{r+1})\in P^\alpha, i_r = i_{r+1}} \left({v^k}_{i_r}(t_r) + c^k_{i_r}(t_r)\right) = \sigma^k$, for every dynamic path p^α in the basis.

Step 1: (Test of optimality)

Solve $|K|$subproblems

$$Z_k = \min \sum_{(i_r,i_{r+1})\in P^\alpha, i_r\neq i_{r+1}} \left(w_{i_r,i_{r+1}}(t_r) + c^k_{i_r,i_{r+1}}(t_r)\right) + \sum_{(i_r,i_{r+1})\in P^\alpha, i_r = i_{r+1}} \left({v^k}_{i_r}(t_r) + c^k_{i_r}(t_r)\right)$$

$s.t.$

$k \in K, p \in P^k, \alpha = 0,1,\ldots,T-\tau_p.$

If $Z_k \geq \sigma^k, \forall k \in K$ **then**

The complementary slackness condition (4) is satisfied. So, the current basis is optimal.

Else if

There is some commodity k, and nonbasic dynamic shortest path Q (with the departure time μ) with $Z_k < \sigma^k$, then, $C^{w,\sigma,v}_{Q^\mu} < 0$. Generate a column corresponding to the dynamic path Q with the departure time μ to the revised simplex method.

End.

Step 2: (Minimum ratio test)

Run minimum ratio test for selecting one dynamic path to remove from the basis.

Step 3: (Pivoting)

After pivoting, transfer a new set of arc prices $w_{ij}(t)$, commodity prices σ^k, and node-commodity prices $v_i^k(t)$, into Step 1. Repeat from Step 1.

5.1. The general illustrative network

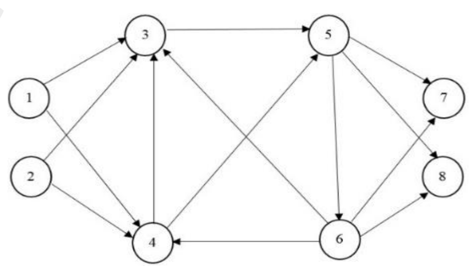

Figure 1. The network in Example 1.

Example 1. We consider a network with 8 nodes and 14 arcs, shown in Figure 1. Suppose that, there are two commodities that should be transmitted over the network. Also, assume that $T = 20$ is the time horizon and $\mathcal{T} = \{1,\ldots,20\}$ and nodes 1 and 7 are the source and the sink nodes for commodity 1, and nodes 2 and

8 are the source and sink nodes for commodity 2, respectively. Also, nominal data are generated randomly using the uniform distribution specified as follows:

1. The cost $c_{ij}^k(t)$ for all $(i,j) \in A, k \in K$, and $t \in \mathcal{T}$, is generated by uniform distribution in the interval $[1, 10]$, and the cost $c_i^k(t)$ for all $i \in N, k \in K$ and $t \in \mathcal{T}$, is generated by uniform distribution in the interval $[1, 5]$.
2. The nominal value for $u_{ij}(t), (i,j) \in A$, and $t \in \mathcal{T}$, is generated by uniform distribution in the interval $[5, 15]$, and the perturbation value $\hat{u}_{ij}(t), (i,j) \in A, t \in \mathcal{T}$, is generated by uniform distribution in the interval $\left[0, u_{ij}(t)\right]$.
3. The nominal value for $u_i^k(t), i \in N, k \in K$, and $t \in \mathcal{T}$, is generated by uniform distribution in the interval $[1, 12]$, and the perturbation value $\hat{u}_i^k(t), i \in N, k \in K$, and $t \in \mathcal{T}$, is generated by uniform distribution in the interval $\left[0, u_i^k(t)\right]$.
4. The nominal value for R_1 is generated by uniform distribution in the interval $[1, 5]$, and the nominal value for R_2 is generated by uniform distribution in the interval $[1, 7]$.
5. The perturbation value for $\hat{R}_1$ is generated by uniform distribution in the interval $[1, R_1]$, and the perturbation value for $\hat{R}_2$ is generated by uniform distribution in the interval $[1, R_2]$,
6. The travel time τ_{ij}, for all $(i,j) \in A$, is generated by uniform distribution in the interval $[1, 3]$.

Table 2. Computational results for SO and DRCC (type II) methods for Example 1.

Method	α	Objective value	CPU time (sec.) Algorithm 1
SO	-	20.1665	614
DRCC (type II)	0.05	20.9732	84
DRCC (type II)	0.10	20.9394	89
DRCC (type II)	0.15	20.8913	94
DRCC (type II)	0.20	20.6291	99
DRCC (type II)	0.25	20.5743	103
DRCC (type II)	0.30	20.4129	115

SO: Stochastic Optimization; DRCC (type II): Distributionally Robust Chance-Constrained in the case of the random data in independent intervals.

Now, we sample 100 scenarios from the independent random variables, and we apply the here-and-now approach in the stochastic optimization techniques. After that, we use model (3.4) for the different values of the confidence parameters to obtain the objective value of the DRCC model. It should be noted that, in this example, the random variables are distributed in the symmetric intervals. For this reason, we can consider the DRCC problem in the case of the independent intervals, denoted by DRCC (type II). The results of this model are summarized in rows 3-8 of Table 2. This table shows that the stochastic optimization technique determines the lowest objective value with the highest CPU time. After that, the DRCC (type II) shows the larger objective values, but its CPU time is much smaller than the SO method. This is the main advantage of the distributionally robust chance-constrained optimization techniques. Figure 2 shows the line graph for the objective values determined by SO and DRCC (type II) (for the different confidence parameters) over the network in Figure 1. This figure shows that, the objective value increases if the confidence parameter in DRCC (type II) increases.

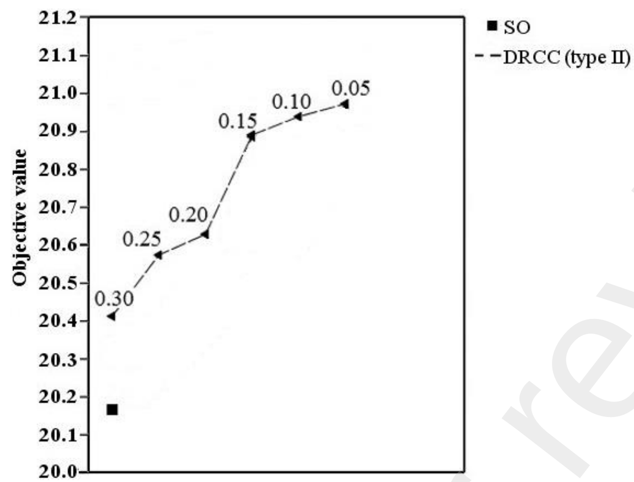

Figure 2. Line graph for SO and DRCC (type II) methods for Example 1.

5.2. Large-scale networks

This section considers two classes of instances with 50 and 100 nodes. For each class, 100 instances were generated (i.e., a total of 200 instances) according to the methodology proposed by Barabasi-Albert [3].

Example 2. This example reports the results obtained by the proposed methods for the network with 50 nodes (see Figure 3). Suppose that 4 commodities should be transmitted over the network. Assume that $T = 20$ is the time horizon and $\mathcal{T} = \{1, \dots, 20\}$ and nodes 1 and 50 are the source and the sink nodes for commodity 1, nodes 2 and 49 are the source and sink nodes for commodity 2, nodes 3 and 48 are the source and sink nodes for commodity 3, nodes 4 and 47 are the source and sink nodes for commodity 4, respectively.

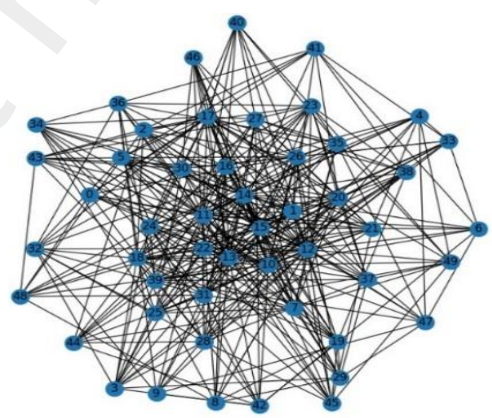

Figure 3. A network instance with 50 nodes is generated by the Barabasi-Albert method.

Table 3. Computational results of SO and DRCC (type II) models for Example 2.

Method	α	Objective value	CPU time (sec.) Algorithm 1
SO	-	78.2615	993
DRCC (type II)	0.05	79.0428	94
DRCC (type II)	0.10	79.0115	147
DRCC (type II)	0.15	78.8918	196
DRCC (type II)	0.20	78.7283	259
DRCC (type II)	0.25	78.5016	273
DRCC (type II)`	0.30	78.4270	328

in Example 1, we use 100 scenarios from the independent random variables and apply the here-and-now approach and report the average objective value and CPU times (on 100 instances) in the second row of Table 3. Then, rows 3-8 of Table 3 report the average objective value and CPU times of the distributionally robust chance-constrained problem (DRCC (type II)), i.e., model (3.4), for the different confidence parameters $\alpha = 0.05, 0.1, 0.15, 0.2, 0.25$ and 0.3. Table 3 also shows that the stochastic optimization technique determines the lower average objective value with the highest average CPU time than the DRCC (type II). In summary, although DRCC (type II) shows the larger average objective values, but its average CPU time is much smaller than the SO model. This is the main advantage of the distributionally robust chance-constrained optimization technique.

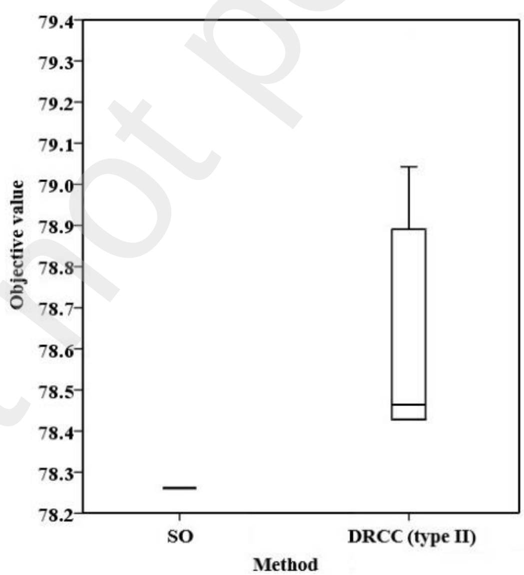

Figure 4. Box-plot for SO and DRCC (type II) method for Example 2.

Figure 4 shows the box-plot for the average objective values, determined by the different methods, i.e., SO and DRCC (type II) (for the different confidence parameters) models, over the network in Figure 3. As we see, SO shows a lower average objective value than the DRCC (type II) method, but Table 3 shows that the average CPU time (100 instances) of the SO method is at least 4.5 times that of the DRCC (type II) which shows the larger objective value than the SO method, in each case of the different confidence parameters.

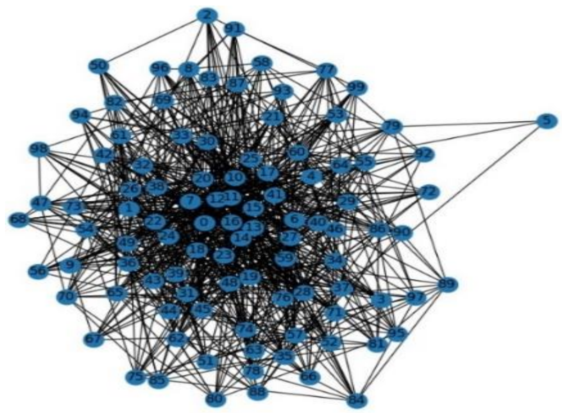

Figure 5. A network instance with 100 nodes is generated by the Barabasi-Albert method.

Example 3. This example reports the results obtained by the proposed methods for the networks with 100 nodes (see Figure 5). Suppose that 4 commodities should be transmitted over the network. Assume that $T = 20$ is the time horizon and $\mathcal{T} = \{1, \dots, 20\}$ and nodes 1 and 100 are the source and sink nodes for commodity 1, nodes 2 and 99 are the source and sink nodes for commodity 2, nodes 3 and 98 are the source and sink nodes for commodity 3, nodes 4 and 97 are the source and sink nodes for commodity 4, respectively. Also, assume that the input parameters are uniformly random generated as follows:

1. The cost $c_{ij}^k(t)$ for all $(i,j) \in A, k \in K$, and $t \in \mathcal{T}$, is generated by uniform distribution in the interval $[1,10]$, and the cost $c_i^k(t)$ for all $i \in N, k \in K$, and $t \in \mathcal{T}$, is generated by uniform distribution in the interval $[1, 5]$.
2. The mean value for $\hat{u}_{ij}(t), (i,j) \in A$, and $t \in \mathcal{T}$, is generated by uniform distribution in the interval $[5, 15]$, and the variance value $\sigma^2_{u_{ij}(t)}, (i,j) \in A, t \in \mathcal{T}$, is generated by uniform distribution in the interval $[0,1]$.
3. The mean value for $\hat{u}_i^k(t), i \in N, k \in K$, and $t \in \mathcal{T}$, is generated by uniform distribution in the interval $[1, 12]$, and the variance value $\sigma^2_{u_i^k(t)}, i \in N, k \in K$, and $t \in \mathcal{T}$, is generated by uniform distribution in the interval $[0, 1]$.
4. The mean value for $\hat{R}_1$ is generated by uniform distribution in the interval $[1, 5]$, the mean value for $\hat{R}_2$is generated by uniform distribution in the interval $[1, 7]$, the variance value $\sigma^2_{R_1}$ is generated by uniform distribution in the interval $[0, 1]$, and the variance value $\sigma^2_{R_2}$ is generated by uniform distribution in the interval $[0, 1]$.
5. The travel time τ_{ij}, for all $(i,j) \in A$, is generated by uniform distribution in the interval $[1, 3]$.

As the previous examples, we use 100 scenarios from the independent random variables and apply the here-and-now approach and report the average objective value and CPU time (on 200 instances) in the second row of Table 4. This example considers the uncertainty set of each parameter as an ellipsoidal, and so, we can apply the DRCC (type I), and DRCC (type III) to determine the average objective value of the minimum cost multicommodity network flow problem. For this purpose, model (3.3) is solved, and the results for the different values of the confidence parameters are summarized in rows 3-8 of Table 4. Then, model (3.6) is used to determine the average objective value of the minimum cost multicommodity network flow, for the different values of the confidence parameters, and the results are reported in rows 9-14 of Table 4. This table shows that the SO method obtains a lower average objective value than the other methods, but the average CPU time of this model is considerable. As we see, the distributionally robust chance-constrained models report the larger average objective values than the stochastic optimization method, but their average CPU time is at least 0.16 of the CPU time of the SO method. Hence, the distributionally robust chance-constrained models are much faster than the stochastic optimization model, and this is the main advantage of the proposed models.

To evaluate the effectiveness of Algorithm 1 for solving the different classes of DRCC and the here-and-now problems, its average computational time has been compared with the average computational time of the LP solver in CPLEX. From Table 4, Algorithm 1 performs faster than the LP solver in CPLEX. In the last column and four rows of Table 4, abbreviation "NI" explains that none of the instances of this class were solved within the time limit of 2000 seconds. It is worth noting that we report the average computational time of LP solver CPLEX just for the instances that were solved up to optimality within the time limit. Also, column 6 calculates the percentage of instances for each class (on 100 instances) that passed the time limit of 2000 seconds for the LP solver CPLEX ("Timeout (%)" in Table 4). The average percentage of unsolved instances, the total of 1300 instances, by LP solver CPLEX is 58.80%.

Table 4. Computational results for SO, DRCC (type I), and DRCC (type III) models for Example 3.

Method	α	Objective value	CPU time (sec.) Algorithm 1	CPU time (sec.) LP solver (CPLEX)	Timeout (%) LP solver (CPLEX)
SO	-	55.3114	1866	NI	100
DRCC (type I)	0.05	55.9963	108	826	16
DRCC (type I)	0.10	55.9814	154	1285	54.50
DRCC (type I)	0.15	55.9523	351	1752	62.66
DRCC (type I)	0.20	55.8916	396	1632	58.66
DRCC (type I)	0.25	55.7129	425	NI	100
DRCC (type I)	0.30	55.6463	433	NI	100
DRCC (type III)	0.05	56.4578	92	521	0
DRCC (type III)	0.10	56.3333	118	689	12
DRCC (type III)	0.15	56.2441	342	1478	39.33
DRCC (type III)	0.20	56.1800	387	1298	42
DRCC (type III)	0.25	56.0014	413	NI	100
DRCC (type III)	0.30	55.9975	421	1803	79.33

DRCC (type I): Distributionally Robust Chance-Constrained in the presence of the family of distributions with known mean and variance; DRCC (type III): Distributionally Robust Chance-Constrained in the case of the radially symmetric nonincreasing distributions; NI: None of the instances of this class were solved within the time limit 2000 seconds by the CPLEX.

Table 5. Computational results for DRCC (type I) and DRCC (type III) models for the network in Example 3 after increasing the radius of each ellipsoidal.

Method	Percentage increase	Objective value
DRCC (type I)	10%	55.9989
DRCC (type I)	20%	56.0010
DRCC (type I)	30%	56.0048
DRCC (type I)	40%	56.0093
DRCC (type I)	50%	56.0128
DRCC (type III)	10%	56.4813
DRCC (type III)	20%	56.5062
DRCC (type III)	30%	56.5241
DRCC (type III)	40%	56.5319
DRCC (type III)	50%	56.5347

Now, we try to construct a larger uncertainty set for each parameter by increasing the radius of each ellipsoidal. For this purpose, we increase the radius of each ellipsoidal by 10%, 20%, 30%, 40%, and 50% of the initial radius and DRCC (type I), i.e., model (3.3) and DRCC (type III), i.e., model (3.4), are solved, and the results are summarized in Table 5 and Figure 5.

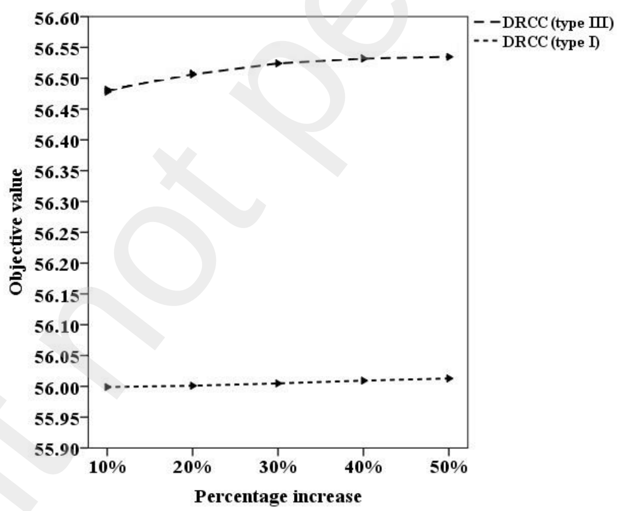

Figure 5. The objective value for DRCC (type I) and DRCC (type III) models after increasing the radius of the ellipsoidal.

6. Conclusions

This study focused on one of the main problems in network literature, namely the multicommodity network flow problem. We investigated the discrete dynamic multicommodity flow (DDMF) for the minimum cost network flow problem with storage at intermediate nodes in the presence of parameter uncertainty. To address parameter uncertainty in the DDMF problem, this paper suggested a new perspective on dealing with the chance constraints in situations where the distribution of random variables is uncertain. We formulated the corresponding distributionally robust chance-constrained (DRCC) optimization model and presented the deterministic restrictions to guarantee the fulfilment of probability constraints. The potential application of the proposed DRCC was illustrated with a numerical example and several experimental tests. The computational results demonstrated that the proposed DRCC requires significantly fewer CPU times than the SO model to solve the uncertain DDMF problem for large-scale networks. Considering that the probability distribution of random variables cannot be precisely determined in many real-world situations, a possible extension of this research would be to explore a different perspective from the robust chance-constrained viewpoint to address optimization problems with randomly distributed uncertain variables. Finally, we compared the average computational times of the proposed algorithm and the LP solver CPLEX. The proposed method outperforms the LP solver CPLEX, especially for large-scale instances.

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