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Group testing with geometry of classical groups over finite fields

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Abstract In this paper, we give an overview of combinatorial group testing and algebra. Our survey focuses on the constructions with algebraic methods, especially geometry of classical groups over finite fields.

Keywords Group testing · Pooling design · Classical groups

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1 Introduction

The basic problem of DNA library screening is to determine which clone (a DNA segment) from the library contains which probe from a given collection of probes in an efficient fashion. A clone is said to be positive for a probe if it contains the probe, and negative otherwise.

This problem is just an instance of the general group testing problem, in which a large population of items containing a small set of defectives are to be tested to identify the defectives efficiently.

Suppose there are n clones including at most d positive ones (others are negative). A (group) test is applicable to an arbitrary subset of clones with two possible outcomes: a

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negative outcome indicates all clones in the subset are negative, and a positive outcome indicates otherwise. A pooling design is a specification of all tests so that they can be performed simultaneously with the goal being to identify all positive clones with a small number of tests. A pooling design M can be represented by a binary incidence matrix where the columns represent clones, the rows represent tests, and $m_{ij} = 1$ if and only if clone j is contained in the subset of test i.

Suppose M has t rows. Then the t outcomes can also be represented by a t-vector $V = (v_1, \ldots, v_t)^T$, where $v_i = 1$ if and only if the outcome of test i is positive ($v_i = 0$ otherwise). Note that V is the Boolean sum of the set of positive clones. Therefore, it is convenient to view a column vector C as a subset S of the base set $\{1, 2, \ldots, t\}$, where $i \in S$ if and only if C has an 1 entry in row i. Then, we can say that V is the union of the set of positive clones. M is called *d-disjunct* if no union of any d columns covers another column. A d-disjunct matrix not only identifies the up-to-d positive clones, but it does so with a simple decoding. Namely, a clone is positive if and only if it (as a column) is contained by V. This is because a negative clone (column) has at least one row not covered by the union of the up-to-d positive clones; such a row then has a negative outcome which identifies the clone as negative.

The notion of d-disjunctness was first raised by Kautz and Singleton [14] in the study of superimposed codes. It was also studied by Erdös et al. [3] under the name of d-cover-free family in extremal set theory. The d-disjunct matrices have become the most important tool in the construction of deterministic pooling designs. Although many constructions have been proposed, the existence of d-disjunct matrices is still sparse.

Macula [21] proposed a novel way of constructing d-disjunct matrices which uses the containment relation in a structure. More specifically, let $[m] := \{1, 2, \ldots, m\}$ be the base set. Then each of the n columns is labeled by a (distinct) k subset of $[m]$, assuming $n \leq \binom{m}{k}$, and each of the $\binom{m}{d}$ rows is labeled by a (distinct) d-subset of $[m]$, where $d < k < m$; $m_{ij} = 1$ if and only if the label of row i is contained in the label of column j. He proved that M is d-disjunct.

Huang and Weng [12] generalized the construction to an arbitrary atomic semilattice where the elements can be ranked. Again, label the columns by a subset of the rank k elements, and label the rows by all rank d elements, $d < k$, and then M is d-disjunct.

Ngo and Du [27] further extended the construction to some geometric structures, such as simplicial complexes, and some graph properties, such as matchings. It is safe to say the ‘containment matrix’ method has opened a new door for constructing d-disjunct matrices from many mathematical structures. However, the basic result in all these constructions is invariably that, to obtain a d-disjunct matrix, use all rank d elements for rows.

One practical problem with this type of construction is that a large n forces d to be large. Then the number of tests could be too large as there are too many rank d elements. This led Macula [23] to propose using the rank 2 elements for rows, regardless of the real d. He showed that while there is no guarantee all positive clones will be identified, the probability of success is still satisfactory when d does not deviate too much from 2. Ngo and Du made a similar comment.

D’Yachkov et al. [2] showed that the containment matrix which uses rank r of elements for rows has the degree d of disjunctness, where r can be much less than d.

In fact, r can be any number from 1 to $k-1$ (k is the lever for columns), while $d \leq q^r$ for some constant q. This is the first happy surprise.

Note that geometric lattices are among pooling spaces. Huang et al. [13] attempted to draw possible connections from finite geometry and distance-regular graphs to pooling spaces: including the projective spaces, the affine spaces, the attenuated spaces, and a few families of geometric lattices associated with the orbits of subspaces under finite classical groups, and associated with d-bounded distance-regular graphs.

Guo et al. [7] introduced the concept of pooling semilattices and proved that a pooling semilattice is a pooling space and then showed how to construct pooling designs from a pooling semilattice. Moreover, they gave many examples of pooling semilattices and thus obtained the corresponding pooling designs.

Guo and Wang [5] gave a new module of pooling design. More specifically, for positive integers $k \leq n$, let $\binom{[n]}{k}$ be the set of all k-subsets of $[n]$. Given integers $1 \leq d < k < n$ and $0 \leq i \leq d$. Let $M(i; d, k, n)$ be the binary matrix with rows indexed with $\binom{[n]}{d}$ and columns indexed with $\binom{[n]}{k}$ such that $M(A, B) = 1$ if and only if $|A \cap B| = i$. It is the first module that is not a containment relation. For more information on pooling designs see the monograph by Du and Hwang [1].

In this paper, we give an overview of constructions on pooling design with geometry of classical groups over finite fields.

The rest of the paper is organized as follows. Section 2 introduces geometry of classical groups over finite fields. Section 3 gives the constructions of pooling designs based on geometry of classical groups over finite fields.

2 Preliminary

In this section, we recall the geometry of classical groups over finite fields.

Let $\mathbb{F}_q$ be a finite field with q elements, where q is a prime power, $\mathbb{F}_q^{(n)}$ be the n-dimensional row vector space over $\mathbb{F}_q$, and $GL_n(\mathbb{F}_q)$ be the *general linear group* of degree n over $\mathbb{F}_q$. $GL_n(\mathbb{F}_q)$ acts on $\mathbb{F}_q^{(n)}$ in the following way:

$$\begin{aligned} &\mathbb{F}_q^{(n)} \times GL_n(\mathbb{F}_q) \to \mathbb{F}_q^{(n)}, \\ &((x_1, x_2, \cdots, x_n), T) \mapsto (x_1, x_2, \cdots, x_n)T. \end{aligned} \tag{1}$$

Let P be an m-dimensional subspace of $\mathbb{F}_q^{(n)}$ and $v_1, v_2, \cdots, v_m$ be a basis of P, then

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} \tag{2}$$

is an $m \times n$ matrix over $\mathbb{F}_q$ of rank m. We call the matrix (2) a *matrix representation* of the subspace P and use also the same letter P to denote the matrix (2) if no ambiguity arises. The action (1) of $GL_n(\mathbb{F}_q)$ on $\mathbb{F}_q^{(n)}$ induces an action on the set of subspaces of $\mathbb{F}_q^{(n)}$ such that $T \in GL_n(\mathbb{F}_q)$ carries the subspace P into PT.

Use any one of the other classical groups, such as the singular general linear group $GL_{n+l,n}(\mathbb{F}_q)$, the symplectic group $S_{p_n}(\mathbb{F}_q)$ (where $n = 2\nu$), the pseudo-symplectic group $Ps_{2\nu+\delta}(\mathbb{F}_q)$, the unitary group $U_n(\mathbb{F}_q)$ (where q is a square), the orthogonal group $O_n(\mathbb{F}_q)$ (where $n = 2\nu + \delta$ and $\delta = 0, 1,$ or 2) and affine-symplectic group $\mathrm{ASp}_{2\nu}(\mathbb{F}_q)$ to replace $GL_n(\mathbb{F}_q)$. Then we can obtain the corresponding geometry.

Now let us introduce the definition of the other classical groups and their corresponding geometries.

- For two nonnegative integers n and l, $\mathbb{F}_q^{(n+l)}$ denotes the $(n+l)$-dimensional row vector space over $\mathbb{F}_q$. The set of all $(n+l) \times (n+l)$ nonsingular matrices over $\mathbb{F}_q$ of the form

$$\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix},$$

where T_{11} and T_{22} are nonsingular $n \times n$ and $l \times l$ matrices, respectively, forms a group under matrix multiplication, called the singular general linear group of degree $n + l$ over $\mathbb{F}_q$ and denoted by $GL_{n+l,n}(\mathbb{F}_q)$. If $l = 0$ (resp. $n = 0$), $GL_{n,n}(\mathbb{F}_q) = GL_n(\mathbb{F}_q)$ (resp. $GL_{l,0}(\mathbb{F}_q) = GL_l(\mathbb{F}_q)$) is the general linear group of degree n (resp. l).

The vector space $\mathbb{F}_q^{(n+l)}$ together with the general linear group action is called the $(n+l)$-dimensional singular linear space over $\mathbb{F}_q$. For $1 \le i \le n+l$, let e_i be the row vector in $\mathbb{F}_q^{(n+l)}$ whose i-th coordinate is 1 and all other coordinates are 0. Denote by E the l-dimensional subspace of $\mathbb{F}_q^{(n+l)}$ generated by $e_{n+1}, e_{n+2}, \ldots, e_{n+l}$. An m-dimensional subspace P of $\mathbb{F}_q^{(n+l)}$ is called a subspace of type (m, k) if $\dim(P \cap E) = k$. The collection of all the subspaces of types $(m, 0)$ in $\mathbb{F}_q^{(n+l)}$, where $0 \le m \le n$, is the attenuated space.

For a fixed subspace P of type (m, k) in $\mathbb{F}_q^{(n+l)}$, let $\mathcal{M}(m_1, k_1; m, k; n+l, n)$ denote the set of all the subspaces of type (m_1, k_1) contained in P , and let $N(m_1, k_1; m, k; n+l, n) = |\mathcal{M}(m_1, k_1; m, k; n+l, n)|$. Then from [31],

$$N(m_1, k_1; m, k; n+l, n) = q^{(m_1-k_1)(k-k_1)} \begin{bmatrix} m-k \\ m_1-k_1 \end{bmatrix}_q \begin{bmatrix} k \\ k_1 \end{bmatrix}_q \tag{3}$$

- Let $n = 2\nu$. It is well known that the cogredience normal form of $2\nu \times 2\nu$ nonsingular alternate matrices is

$$K = \begin{pmatrix} 0 & I^{(\nu)} \\ -I^{(\nu)} & 0 \end{pmatrix}.$$

Let

$$S_{p_{2\nu}}(\mathbb{F}_q) = \{T \in GL_{2\nu}(\mathbb{F}_q) | TKT^{\mathrm{T}} = K\}.$$

Then $S_{p_{2\nu}}(\mathbb{F}_q)$ is a group with respect to the matrix multiplication, called the *symplectic group* of degree 2ν over $\mathbb{F}_q$.

The vector space $\mathbb{F}_q^{(2\nu)}$ together with the symplectic group action is called the 2ν-dimensional space over $\mathbb{F}_q$. Let P be an m-dimensional subspace of $\mathbb{F}_q^{(2\nu)}$. An m-dimensional subspace P is said to be of type (m, r), if PKP^{T} is of rank $2r$. In particular, subspaces of type $(m, 0)$ are called m-dimensional *totally isotropic subspaces.*The subspaces of type (m, r) exist if and only if $2r \le m \le \nu + r$. The subspace of type (m, r), which contains subspaces of type (m_1, r), exists if and only if $2r \le m_1 \le m \le \nu + r$.

Let $N(m_1, r; m, r; 2\nu)$ denote the number of subspaces of type (m_1, r) contained in a given subspace of type (m, r). It is known that

$$N(m_1, r; m, r; 2\nu) = q^{2r(m-m_1)} \frac{\prod_{i=m-m_1+1}^{m-2r}(q^i - 1)}{\prod_{i=1}^{m_1-2r}(q^i - 1)}. \tag{4}$$

- Let $\mathbb{F}_q$ be a finite field of characteristic 2, and let

$$S_1 = \begin{pmatrix} 0 & I^{(\nu)} & \\ I^{(\nu)} & 0 & \\ & & 1 \end{pmatrix}, S_2 = \begin{pmatrix} 0 & I^{(\nu)} & & \\ I^{(\nu)} & 0 & & \\ & & 0 & 1 \\ & & 1 & 1 \end{pmatrix}.$$

The *pseudo-symplectic group* of degree $2\nu + \delta (\delta = 1, \text{or } 2)$ over $\mathbb{F}_q$, denoted by $Ps_{2\nu+\delta}(\mathbb{F}_q)$, consists of all $(2\nu+\delta)\times(2\nu+\delta)$ matrix T over $\mathbb{F}_q$ satisfying $TS_\delta T^{\mathrm{T}} = S_\delta$.

The vector space $\mathbb{F}_q^{(2\nu+\delta)}$ together with the pseudo-symplectic group action is called the $(2\nu + \delta)$-*dimensional pseudo-symplectic space over* $\mathbb{F}_q$ of characteristic 2.

Let P be an m-dimensional subspace of $\mathbb{F}_q^{(2\nu+\delta)}$. $PS_\delta P^{\mathrm{T}}$ is cogredient to one of the following three forms

$$M(m, 2r, r) = \begin{pmatrix} 0 & I^{(r)} & \\ I^{(r)} & 0 & \\ & & 0^{(m-2r)} \end{pmatrix},$$

$$M(m, 2r+1, r) = \begin{pmatrix} 0 & I^{(r)} & & \\ I^{(r)} & 0 & & \\ & & 1 & \\ & & & 0^{(m-2r-1)} \end{pmatrix},$$

and

$$M(m, 2r+2, r) = \begin{pmatrix} 0 & I^{(r)} & & & \\ I^{(r)} & 0 & & & \\ & & 0 & 1 & \\ & & 1 & 1 & \\ & & & & 0^{(m-2r-2)} \end{pmatrix}$$

for some r such that $0 \le r \le \lfloor \frac{m}{2} \rfloor$. We say that P is a *subspace of type* $(m, 2r+\tau, r, \varepsilon)$, where $\tau = 0, 1$, or 2 and $\varepsilon = 0$, or 1, if

(i) $PS_\delta P^{\mathrm{T}}$ is cogredient to $M(m, 2r+\tau, r)$, and
(ii) $e_{2\nu+1} \notin P$ or $e_{2\nu+1} \in P$ according to $\varepsilon = 0$ or 1, respectively.

In particular, subspaces of type $(m, 0, 0, 0)$ and $(m, 0, 0, 1)$ are called m-dimensional *totally isotropic subspaces*. The subspaces of type $(m, 2r+1, r, 1)$ exist if and only if $2r+1 \leq m \leq \nu+r+1$. The subspaces of type $(m, 2r+1, r, 1)$, which contain subspaces of type $(m_1, 2r+1, r, 1)$, exist if and only if $2r+1 \leq m_1 < m \leq \nu+r+1$.

Let $N(m_1, 0, 0, 0; m, 0, 0, 0; 2\nu+\delta)$ denote the number of subspaces of type $(m_1, 0, 0, 0)$ contained in a given subspace of type $(m, 0, 0, 0)$. From [30],

$$N(m_1, 0, 0, 0; m, 0, 0, 0; 2\nu+\delta) = \frac{\prod_{i=m-m_1+1}^{m}(q^i-1)}{\prod_{i=1}^{m_1}(q^i-1)}. \tag{5}$$

- Let $q = q_0^2$, where q_0 is a prime power. $\mathbb{F}_q = \mathbb{F}_{q_0^2}$ has an involutive automorphism

$$- : a \to \bar{a},$$

whose fixed field is $\mathbb{F}_{q_0}$. Let

$$U_n(\mathbb{F}_q) = \{T \in GL_n(\mathbb{F}_q) | T\bar{T}^{\mathrm{T}} = I^{(n)}\}.$$

Then $U_n(\mathbb{F}_q)$ is a group with respect to the matrix multiplication, called the *unitary group* of degree n over $\mathbb{F}_q$.

The vector space $\mathbb{F}_q^{(n)}$ together with the unitary group action is called the n-*dimensional unitary space over* $\mathbb{F}_q$. An m-dimensional subspace P is said to be of type (m, r), if $PH_\delta(\bar{P})^{\mathrm{T}}$ is of rank r. In particular, subspaces of type $(m, 0)$ are called m-dimensional *totally isotropic subspaces.*The subspaces of type (m, r) exist if and only if $2r \leq 2m \leq n+r$. The subspace of type (m, r), which contains subspaces of type (m_1, r_1), exists if and only if $2r \leq 2m \leq n+r$, $2r_1 \leq 2m_1 \leq n+r_1$ and $0 \leq r-r_1 \leq 2(m-m_1)$. Let $N(m_1, r; m, r; n)$ denote the number of subspaces of type (m_1, r) contained in a given subspace of type (m, r). From [30]

$$N(m_1, r; m, r; n) = q^{2r(m-m_1)} \frac{\prod_{i=m-m_1+1}^{m-r}(q^{2i}-1)}{\prod_{i=1}^{m_1-r}(q^{2i}-1)}. \tag{6}$$

- Let $\mathbb{F}_q$ be a finite field of odd characteristic. For a fixed non-square element z of $\mathbb{F}_q^*$, let

$$S_{2r+\delta,\,\Delta} = \begin{pmatrix} 0 & I^{(\nu)} & \\ I^{(\nu)} & 0 & \\ & & \Delta \end{pmatrix}, \text{ where } \Delta = \begin{cases} \emptyset, & \text{if } \delta = 0, \\ (1) \text{ or } (z), & \text{if } \delta = 1, \\ \mathrm{diag}(1, -z), & \text{if } \delta = 2. \end{cases}$$

The *orthogonal group* of degree $2\nu+\delta$ over $\mathbb{F}_q$, denoted by $O_{2\nu+\delta,\Delta}(\mathbb{F}_q)$, consists of all $(2\nu+\delta)\times(2\nu+\delta)$ matrix T over $\mathbb{F}_q$ satisfying $TS_{2\nu+\delta,\Delta}T^{\mathrm{T}} = S_{2\nu+\delta,\Delta}$.

The vector space $\mathbb{F}_q^{(2\nu+\delta)}$ together with the orthogonal group action is called the $(2\nu+\delta)$-*dimensional orthogonal space over* $\mathbb{F}_q$ of odd characteristic.

Let P be an m-dimensional subspace of $\mathbb{F}_q^{(2\nu+\delta)}$. $PS_{2\nu+\delta}P^{\mathrm{T}}$ is cogredient to one of the following four forms

$$M(m,2r,r)=\begin{pmatrix}0 & I^{(r)} & \\ I^{(r)} & 0 & \\ & & 0^{(m-2r)}\end{pmatrix},$$

$$M(m,2r+1,r,1)=\begin{pmatrix}0 & I^{(r)} & & \\ I^{(r)} & 0 & & \\ & & 1 & \\ & & & 0^{(m-2r-1)}\end{pmatrix},$$

$$M(m,2r+1,r,z)=\begin{pmatrix}0 & I^{(r)} & & \\ I^{(r)} & 0 & & \\ & & z & \\ & & & 0^{(m-2r-1)}\end{pmatrix},$$

$$M(m,2r+2,r)=\begin{pmatrix}0 & I^{(r)} & & & \\ I^{(r)} & 0 & & & \\ & & 1 & & \\ & & & -z & \\ & & & & 0^{(m-2r-2)}\end{pmatrix}.$$

We say that P is a *subspace of type* $(m,2r+\gamma,r,\Gamma)$, if $PS_{2\nu+\delta}P^{\mathrm{T}}$ is cogredient to $M(m,2r+\gamma,r,\Gamma)$, where $\Gamma=\emptyset$, if $\gamma=0$, and $\Gamma=0$ or (z), if $\gamma=1$, and $\Gamma=\begin{pmatrix}1 & \\ & -z\end{pmatrix}$, if $\gamma=2$. In particular, subspaces of type $(m,0,0)$ are called m-dimensional *totally isotropic subspaces*. The subspaces of type $(m,2r+1,r,1)$ exist if and only if $2r+1\le m\le \nu+r+1$. The subspace of type $(m,2r+1,r,1)$, which contains subspaces of type $(m_1,2r+1,r,1)$, exists if and only if $2r+1\le m_1 < m\le \nu+r+1$. Let $N(m_1,0,0;m,0,0;2\nu+\delta,\Delta)$ denote the number of subspaces of type $(m_1,0,0)$ contained in a given subspace of type $(m,0,0)$. From [30],

$$N(m_1,0,0;m,0,0;2\nu+\delta,\Delta)=\frac{\prod_{i=m-m_1+1}^{m}(q^i-1)}{\prod_{i=1}^{m_1}(q^i-1)}. \tag{7}$$

- Let $\mathbb{F}_q$ be a finite field of characteristic 2. Denote by $\mathcal{K}_n$ the set of all $n\times n$ alternate matrices over F_q. Two $n\times n$ matrices A and B over F_q are said to be *congruent* mod $\mathcal{K}_n$, denoted $A\equiv B$ (mod $\mathcal{K}_n$), if $A-B\in\mathcal{K}_n$. Clearly, $\equiv$ is an equivalence relation on the set of all $n\times n$ matrices. Let $[A]$ denote the equivalence class containing A. Two matrix classes $[A]$ and $[B]$ are said to be *cogredient* if there is a nonsingular $n\times n$ matrix Q over F_q such that $[QAQ^{\mathrm{T}}]\equiv[B]$.

Let

$$G_{2\nu+\delta,\,\Delta} = \begin{pmatrix} 0 & I^{(\nu)} & \\ & 0 & \\ & & \Delta \end{pmatrix}, \text{ where } \Delta = \begin{cases} \emptyset, & \text{if } \delta = 0, \\ (1), & \text{if } \delta = 1, \\ \begin{pmatrix} \alpha & 1 \\ & \alpha \end{pmatrix}, & \text{if } \delta = 2, \end{cases}$$

where α is a fixed element of F_q such that $\alpha \notin \{x^2 + x | x \in F_q\}$.

The orthogonal group of degree $2\nu + \delta$ over F_q with respect to $G_{2\nu+\delta,\,\Delta}$, denoted by $O_{2\nu+\delta,\,\Delta}(F_q)$, consists of all $(2\nu + \delta) \times (2\nu + \delta)$ matrices T over F_q satisfying $[T G_{2\nu+\delta,\,\Delta} T^{\mathrm{T}}] \equiv [G_{2\nu+\delta,\,\Delta}]$.

The vector space $\mathbb{F}_q^{(2\nu+\delta)}$ together with the orthogonal group action is called the $(2\nu + \delta)$-*dimensional orthogonal space over* $\mathbb{F}_q$ of characteristic 2.

Let P be an m-dimensional subspace of $\mathbb{F}_q^{(2\nu+\delta)}$. $P G_{2\nu+\delta} P^{\mathrm{T}}$ is *cogredient* to one of the following three forms

$$M(m, 2r, r) = \begin{pmatrix} 0 & I^{(r)} & \\ & 0 & \\ & & 0^{(m-2r)} \end{pmatrix},$$

$$M(m, 2r+1, r) = \begin{pmatrix} 0 & I^{(r)} & & \\ & 0 & & \\ & & 0^{(m-2r-1)} & \\ & & & 1 \end{pmatrix},$$

and

$$M(m, 2r+2, r) = \begin{pmatrix} 0 & I^{(r)} & & & \\ & 0 & & & \\ & & \alpha & 1 & \\ & & & \alpha & \\ & & & & 0^{(m-2r-2)} \end{pmatrix}.$$

We say that P is a *subspace of type* $(m, 2r + \gamma, r, \Gamma)$, if $P G_{2\nu+\delta} P^{\mathrm{T}}$ is *cogredient* to $M(m, 2r + \gamma, r)$, where $\Gamma = 1$ or 0, if $e_{2\nu+1} \in P$ or not , respectively, in case $\delta = \gamma = 1$. In particular, subspaces of type $(m, 0, 0)$ are called m-dimensional *totally singular subspaces.* The subspaces of type $(m, 2r + 1, r, 1)$ exist if and only if $2r + 1 \le m \le \nu + r + 1$. The subspace of type $(m, 2r + 1, r, 1)$, which contains subspaces of type $(m_1, 2r+1, r, 1)$, exists if and only if $2r+1 \le m_1 < m \le \nu+r+1$. Let $N(m_1, 0, 0; m, 0, 0; 2\nu + \delta)$ denote the number of subspaces of type $(m_1, 0, 0)$ contained in a given subspace of type $(m, 0, 0)$. From [30],

$$N(m_1, 0, 0; m, 0, 0; 2\nu + \delta) = \frac{\prod_{i=m-m_1+1}^{m} (q^i - 1)}{\prod_{i=1}^{m_1} (q^i - 1)}. \quad (8)$$

- Suppose P is a subspace of type (m, r) in 2ν dimension symplectic space $\mathbb{F}_q^{(2\nu)}$. A coset of $\mathbb{F}_q^{(2\nu)}$ relative to a subspace P of type (m, r) is called a (m, r)-*flat*. A flat F_1 is said to be *incident* with a flat F_2, if F_1 contains or is contained in F_2. The point set $\mathbb{F}_q^{(2\nu)}$ with all the flats and the incidence relation among them defined above is said to be the 2ν-dimensional *affine-symplectic* space, denoted by $\mathrm{ASG}(2\nu, \mathbb{F}_q)$.

The set of matrices of the form

$$\begin{pmatrix} T & 0 \\ v & 1 \end{pmatrix},$$

where $T \in Sp_{2\nu}(\mathbb{F}_q)$ and $v \in \mathbb{F}_q^{(2\nu)}$, forms a group under matrix multiplication. This group is said to be the *affine-symplectic* group of $\mathrm{ASG}(2\nu, \mathbb{F}_q)$, denoted by $\mathrm{ASp}_{2\nu}(\mathbb{F}_q)$. Define the action of $\mathrm{ASp}_{2\nu}(\mathbb{F}_q)$ on the $\mathrm{ASG}(2\nu, \mathbb{F}_q)$ as follows:

$$\begin{array}{ccc} \mathrm{ASG}(2\nu, \mathbb{F}_q) \times \mathrm{ASp}_{2\nu}(\mathbb{F}_q) & \to & \mathrm{ASG}(2\nu, \mathbb{F}_q) \\ \left(x, \begin{pmatrix} T & 0 \\ v & 1 \end{pmatrix}\right) & \mapsto & xT + v. \end{array}$$

Then affine-symplectic group $\mathrm{ASp}_{2\nu}(\mathbb{F}_q)$ acts transitive on the set of (m, r)-flats in $\mathrm{ASG}(2\nu, \mathbb{F}_q)$[30].

For more information about geometry of classical groups over finite fields, see [30]

3 The constructions

3.1 The constructions based on the vector space

3.1.1 Use subspaces of the finite vector space

Consider the m-dimensional space, or simply m-space, of $\mathbb{F}_q^{(n)}$ where q is a prime or a prime power. Let $\begin{bmatrix} m \\ k \end{bmatrix}_q$ denote the number of k-dimensional subspaces, or simply k-space. It is well known [29] (p. 291) that the following is true.

Lemma 3.1

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = \frac{(q^m - 1)(q^{m-1} - 1) \cdots (q^{m-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}$$

and

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = \begin{bmatrix} m \\ m-k \end{bmatrix}_q.$$

Definition 3.2 Fix integers $1 \le r < k < m$. Let $M(m, k, r)$ be the 01-matrix by taking all k-spaces (from an underlying m-space) as columns and all r-spaces as rows. $M(m, k, r)$ has a 1 in row i and column j if and only if i is contained in j.

$M(m, k, r)$ was first studied by Yakir [32] from a linear algebra point of view and by Ngo and Du [27] from a pooling design point of view. $M(m, k, r)$ is easily checked to be a ranked atomic semilattice, thus the matrix is r-disjunct and hence (Huang and Weng [12]) d^z-disjunct for some $1 \leq d \leq r$ and

$$z = \begin{bmatrix} k-d \\ r-d \end{bmatrix}_q .$$

Note that the construction still requires the row rank r being at least as large as the upper bound d of the number of positive clones. D'yachkov et al. obtained the following theorem.

Theorem 3.3 [2] *Suppose* $k - r \geq 2$ *and set* $p := \frac{q(q^{k-1}-1)}{q^{k-r}-1}$. *Then* $M(m, k, r)$ *is* d^z*-disjunct for* $1 \leq d \leq p$ *and*

$$z = q^{k-r} \begin{bmatrix} k-1 \\ r-1 \end{bmatrix}_q - (d-1)q^{k-r-1} \begin{bmatrix} k-2 \\ r-1 \end{bmatrix}_q .$$

Nan and Guo [25] generalized Ngo and Du's construction [27] and obtained a family of pooling designs

Let $\mathcal{M}(m, n)$ be the set of all m-dimensional subspaces of $\mathbb{F}_q^{(n)}$.

Definition 3.4 [25] Given integers $1 \leq r, m \leq n-1$ and $\max\{0, r+m-n\} \leq j \leq \min\{r, m\}$. Let $M(r, m; n)$ be the binary matrix whose rows (resp. columns) are indexed by $\mathcal{M}(r, n)$(resp. $\mathcal{M}(m, n)$). We also order elements of these sets lexicographically. $M(r, m; n)$ has a 1 in row i and column l if and only if the i-th subspace of $\mathcal{M}(r, n)$ intersect the l-th subspace of $\mathcal{M}(m, n)$ at j-dimensional subspaces of $\mathbb{F}_q^{(n)}$.

Then $M(r, m; n)$ is an $\begin{bmatrix} n \\ r \end{bmatrix}_q \times \begin{bmatrix} n \\ m \end{bmatrix}_q$ matrix, whose constant row (resp. column) weight is $p^r_{j,j}(r, m; n) = q^{(r-j)(m-j)} \begin{bmatrix} n-r \\ m-j \end{bmatrix}_q \begin{bmatrix} r \\ j \end{bmatrix}_q$ (resp. $p^m_{j,j}(m, r; n) = q^{(r-j)(m-j)} \begin{bmatrix} n-m \\ r-j \end{bmatrix}_q \begin{bmatrix} m \\ j \end{bmatrix}_q$).

Theorem 3.5 [25] *Let* $1 \leq r, m \leq n-1$ *and* $\max\{0, r+m-n\} \leq j \leq \min\{r, m\}$. *If* $1 \leq d \leq \lfloor p^m_{j,j}(m, r; n)/\alpha \rfloor + 1$, *then* $M(r, m; n)$ *is* d^e*-disjunct, where* $e = p^m_{j,j}(m, r; n) - d\alpha - 1$, $\alpha = \max\{p^l_{j,j}(m, r; n) | \max\{0, 2m-n\} \leq l \leq m-1\}$.

The error-tolerance property of $M(r, m; n)$ is not well expressed. Guo and Wang [6] constructed a family pooling designs whose error-tolerance property is better than that of [25].

Definition 3.6 [6] For positive integers $1 \leq d < k < n$ and $\max\{0, d+k-n\} \leq i \leq d$, let $M_q(i; d, k, n)$ be the binary matrix by taking all k-spaces (from an underlying n-space) as columns and all d-spaces as rows such that $M_q(A, B) = 1$ if and only if $\dim(A \cap B) = i$.

Theorem 3.7 [6] *Let i, d, k, n be positive integers with $\lfloor (d+1)/2 \rfloor \le i \le d < k$ and $n-k-\bar{s}(k+d-2i) \ge d-i$. If $k-i \ge 2$ and $1 \le \bar{s} \le q(q^{k-1}-1)/(q^{k-i}-1)$, then the following hold:*

(i) *$M_q(i; d, k, n)$ is an $\bar{s}^{\bar{e}_2}$-disjunct matrix, where*

$$\bar{e}_2 = q^{(d-i)(k+\bar{s}(k+d-2i)-i)} \begin{bmatrix} n-k-\bar{s}(k+d-2i) \\ d-i \end{bmatrix}_q \times \left(q^{k-i} \begin{bmatrix} k-1 \\ i-1 \end{bmatrix}_q - (\bar{s}-1) q^{k-i-1} \begin{bmatrix} k-2 \\ i-1 \end{bmatrix}_q \right) - 1;$$

(ii) *for a given k, if $i < d$, then $\lim_{n\to\infty} \frac{\bar{e}_2+1}{\bar{e}_1+1} = \infty$. where $\bar{e}_1 = z-1$, z is as in Theorem 3.3.*

The error-tolerance property of [6] is much better than that of [27] under some conditions.

3.1.2 Use subspaces which join a fixed subspace being the $\mathbb{F}_q^{(n)}$

In 2015, Liu and Gao [20] constructed a family pooling designs. Denote the set of all i-subspaces U of vector space V satisfying $U + W = V$ by $M(i; n, b)$, where W is a fixed $(n-b)$-subspace of $\mathbb{F}_q^{(n)}$.

Definition 3.8 [20] Given integers $1 \le b < l_1 < l_2 < n$. Let $M(l_1, l_2; n, b)$ be the binary matrix whose rows (resp. columns) are indexed by $M(l_1; n, b)$ (resp. $M(l_2; n, b)$). We also order elements of these sets lexicographically. $M(l_1, l_2; n, b)$ has a 1 in row i and column j if and only if the ith subspace of $M(l_1; n, b)$ is a subspace of the jth subspace of $M(l_2; n, b)$.

Theorem 3.9 [20] *Suppose $l_1 > b$, $l_2 - l_1 \ge 2$, and let $p = \frac{\left(\frac{q^{l_2}-q^b}{q^{l_2-l_1}-1}-1\right)\left(q^{l_2-b-1}-1\right)}{\left(\frac{q^{l_2-1}-q^b}{q^{l_2-l_1-1}-1}-1\right)\left(q^{l_2-l_1-1}-1\right)}$. Then $M(l_1, l_2; n, b)$ is d_z-disjunct for $1 \le d \le pq^b$, and $z = (\frac{q^{l_2}-q^b}{q^{l_2-l_1}-1} - 1)q^{b(l_2-l_1-1)} \begin{bmatrix} l_2-b-1 \\ l_1-b \end{bmatrix}_q - (d-1)(\frac{q^{l_2-1}-q^b}{q^{l_2-l_1-1}-1} - 1)q^{b(l_2-l_1-2)} \begin{bmatrix} l_2-b-2 \\ l_1-b \end{bmatrix}_q$. Moreover, if $1 \le d \le \min\{pq^b, q\}$, then $M(l_1, l_2; n, b)$ is full d^z -disjunct.*

The test efficiency of construction is smaller than that of [2] under some conditions.

3.2 Use subspaces of singular linear space

Liu and Gao [19] construct a family of error-correcting pooling designs with the incidence matrix of two types of subspaces of singular linear space over finite fields, and exhibit their disjunct properties.

Definition 3.10 [19] Given integers $0 \le k \le l, 0 \le m-k \le n, 0 \le r \le m-k-2$, Let $M(r; m, k; n+l, n)$ be the binary matrix whose rows (resp. columns) are indexed by $\mathcal{M}(r, 0; n+l, n)$ (resp.$\mathcal{M}(m, k; n+l, n)$. We also order elements of these sets lexicographically. $M(r; m, k; n+l, n)$ has a 1 in row i and column j if and only if the $i-$th subspace of $\mathcal{M}(r, 0; n+l, n)$ is a subspace of the jth subspace of $\mathcal{M}(m, k; n+l, n)$.

Theorem 3.11 [19] *Given integers* $0 \le k < l, 0 \le m-k \le n, 0 \le r \le m-k-2$ *and let* $t = N(r, 0; m, k; n+l, n)$, $u = N(r, 0; m-1, k; n+l, n))$, $v = N(r, 0; m-1, k-1; n+l, n)$, $x = N(r, 0; m-2, k; n+l, n)$, $y = N(r, 0; m-2, k-1; n+l, n)$, $z = N(r, 0; m-2, k-2; n+l, n)$ *and* $w = \max\{u-x, u-y, u-z, v-x, v-y, v-z\}$, *if* $1 \le d \le \lfloor \frac{t-\max\{u,v\}-1}{w} \rfloor + 1$ *then* $M(r; m, k; n+l, n)$ *is* d^e-*disjunct, where* $e = t-\max\{u, v\}-(d-1)w-1$. *In particular, if* $1 \le d \le \min\{\lfloor \frac{t-\max\{u,v\}-1}{w} \rfloor + 1, q+1\}$, *then* $M(r; m, k; n+l, n)$ *is fully* d^e-*disjunct, where* $N(m_1, k_1; m, k; n+l, n)$ *is from* (3).

3.3 The constructions based on the symplectic space

3.3.1 Use subspaces containing a fixed $(m_0, 0)$*-space and contained in its dual space*

In 2008, Zhang et al. [33] constructed a family pooling designs based on the symplectic space.

Definition 3.12 [33] Select integers $0 \le m_0 < r < m \le \nu$. Assume that P_0 is a fixed $(m_0, 0)$-space of $\mathbf{F}_q^{(2\nu)}$. Let M be the $(0, 1)$-matrix by taking all $(m, 0)$-spaces which are contained in $P_0^{\perp}$ and contain P_0 as columns and all $(r, 0)$-spaces which are contained in $P_0^{\perp}$ and contain P_0 as rows. M has a 1 in row i and column j if and only if i is contained in j.

Theorem 3.13 [33] *Suppose* $m - r \ge 2$ *and set* $b = \frac{q(q^{m-m_0-1}-1)}{q^{m-r}-1}$. *Then* M *is* d^z*-disjunct for* $1 \le d \le b$ *and*

$$z = \begin{bmatrix} m-m_0 \\ r-m_0 \end{bmatrix} - d \begin{bmatrix} m-m_0-1 \\ r-m_0 \end{bmatrix} + (d-1) \begin{bmatrix} m-m_0-2 \\ r-m_0 \end{bmatrix}.$$

The test efficiency of construction is smaller than that of [2] under some conditions.

3.3.2 Use subspaces containing a fixed (d_0, r)*-space*

In 2010, Li et al. [16] constructed two family pooling designs based on the symplectic space.

Definition 3.14 [16] For $2r \le d_0 < d < k \le \nu + r$, assume that P_0 is a fixed (d_0, r)-space of $\mathbb{F}_q^{(2\nu)}$. Let M be a binary matrix whose columns (rows) indexed by all (k, r)-spaces containing P_0((d, r)-spaces containing P_0) in $\mathbb{F}_q^{(2\nu)}$ such that $M(A, B) = 1$ if $A \subseteq B$ and 0 otherwise. This matrix is denoted by $M_1(\nu, d, k)$.

Theorem 3.15 [16] *Suppose* $2r \leq d_0 < d < k \leq \nu + r$ *and set* $b = \frac{q(q^{k-d_0-1}-1)}{q^{k-d}-1}$. *Then* $M_1(\nu, d, k)$ *is* s^e*-disjunct for* $1 \leq d \leq b$ *and*

$$
\begin{aligned}
e = q^{k-d} N(d - d_0 - 1, 0; k - d_0 - 1, 0; 2(\nu + r - d_0)) \\
- (s-1)q^{k-d-1} N(d - d_0 - 1, 0; k - d_0 - 2, 0; 2(\nu + r - d_0)),
\end{aligned}
$$

where $N(m_1, r; m, r; 2\nu)$ *is from* (4).

The test efficiency of construction is smaller than that of [2] under some conditions.

Definition 3.16 [16] For $2 \leq 2r \leq d < k \leq \nu + r$, let M be a binary matrix whose columns (rows) indexed by all subspaces of type (k, r) $((d, r))$ in $\mathbb{F}_q^{(2\nu)}$ such that $M(A, B) = 1$ if $A \subseteq B$ and 0 otherwise. This matrix is denoted by $M_2(\nu, d, k)$.

Theorem 3.17 [16] *Suppose* $4 \leq 2r + 2 \leq d < k - 1 \leq \nu + r - 1$. *If* $1 \leq s \leq q^{2r}$, *then* $M_2(\nu, d, k)$ *is* s^e*-disjunct, where* $e = q^{(k-d-1)d+2r}$.

The test efficiency of construction is smaller than that of [2] under some conditions.

3.3.3 Use totally isotropic subspaces and non-totally isotropic subspaces

In 2010, Guo et al. [8] constructed a family of inclusion matrices associated with subspaces of the symplectic space $\mathbb{F}_q^{(2\nu)}$.

Definition 3.18 [8] Given integers $1 \leq r \leq m < \nu$. Let $M(r, 2m; 2\nu)$ be the binary matrix whose rows (resp. columns) are indexed by $M(r, 0; 2\nu)$ (resp. $M(2m, m; 2\nu)$). We also order elements of these sets lexicographically. $M(r, 2m; 2\nu)$ has a 1 in row i and column j if and only if the i-th subspace of $M(r, 0; 2\nu)$ is a subspace of the j-th subspace of $M(2m, m; 2\nu)$.

Theorem 3.19 [8] *Let* $1 \leq r \leq m < \nu$, *and let* $\beta = N(r, 0; 2m, m; 2\nu)$, $\gamma = N(r, 0; 2m - 1, m - 1; 2\nu)$, $\delta = N(r, 0; 2m - 2, m - 1; 2\nu)$, $\xi = N(r, 0; 2m - 2, m - 2; 2\nu)$ *and* $\alpha = \max\{\gamma - \delta, \gamma - \xi\}$, *where* $N(m_1, r; m, r; 2\nu)$ *is from* (4). *Then the following (i)–(iii) hold:*

(i) *For* $m \geq 2$ *and* $m \geq r + 1$, *if* $1 \leq d \leq \left\lfloor \frac{\beta - \gamma - 1}{\alpha} \right\rfloor + 1$, *then* $M(r, 2m; 2\nu)$ *is* d^e*-disjunct, where* $e = \beta - \gamma - (d - 1)\alpha - 1$. *In particular, if* $1 \leq d \leq \min\{\left\lfloor \frac{\beta - \gamma - 1}{\alpha} \right\rfloor + 1, q + 1\}$, *then* $M(r, 2m; 2\nu)$ *is fully* d^e*-disjunct.*

(ii) *For* $m \geq 2$ *and* $m = r$, *if* $1 \leq d \leq \left\lfloor \frac{\beta - 1}{\gamma} \right\rfloor$, *then* $M(r, 2m; 2\nu)$ *is* d^e*-disjunct, where* $e = \beta - d\gamma - 1$. *In particular, if* $1 \leq d \leq \min\{\left\lfloor \frac{\beta - 1}{\gamma} \right\rfloor, q + 1\}$, *then* $M(r, 2m; 2\nu)$ *is fully* d^e*-disjunct.*

(iii) *For* $m = 1$, *if* $1 \leq d \leq q$, *then* $M(1, 2; 2\nu)$ *is fully* d^e*-disjunct, where* $e = q - d$.

The test efficiency of construction is smaller than that of [33] under some conditions.

3.3.4 Use totally isotropic subspaces

In 2014, Guo and Nan [11] generalized [8]. They constructed a new family pooling designs based on the symplectic space.

Definition 3.20 [11] Given integers $1 \leq r \leq \left\lfloor \frac{m}{2} \right\rfloor$, $2s \leq m \leq \nu + s$ and $s + r \leq m < 2\nu$. Let $M(r, 0; m, s; 2\nu)$ be the binary matrix whose rows (resp. columns) are indexed by $M(r, 0; 2\nu)$ (resp. $M(m, s; 2\nu)$). $M(r, 0; m, s; 2\nu)$ has a 1 in row i and column j, if and only if the ith subspace of type $(r, 0)$ is contained in the jth subspace of type (m, s).

Theorem 3.21 [11] *Let* $1 \leq r \leq \left\lfloor \frac{m}{2} \right\rfloor$, $s + r \leq m < 2\nu$, *and let* $\beta = N(r, 0; m, s; 2\nu)$, $\gamma = N(r, 0; m-1, s; 2\nu)$, $\xi = N(r, 0; m-1, s-1; 2\nu)$, $\eta = N(r, 0; m-2, s; 2\nu)$, $\zeta = N(r, 0; m-2, s-1; 2\nu)$, $\delta = N(r, 0; m-2, s-2; 2\nu)$, $\alpha = \max\{\gamma-\eta, \gamma-\zeta, \gamma-\delta, \xi-\eta, \xi-\zeta, \xi-\delta\}$, *where* $N(m_1, r; m, r; 2\nu)$ *is from* (4). *If* $d \leq \left\lfloor \frac{\beta-\max\{\gamma,\xi\}-1}{\alpha} \right\rfloor + 1$, *then* $M(r, 0; m, s; 2\nu)$ *is* d^e*-disjunct, where* $e = \beta-\max\{\gamma, \xi\}-(d-1)\alpha-1$. *In particular, if* $1 \leq d \leq \min\{\left\lfloor \frac{\beta-\max\{\gamma,\xi\}-1}{\alpha} \right\rfloor+1, q+1\}$, *then* $M(r, 0; m, s; 2\nu)$ *is fully* d^e*-disjunct.*

The error-tolerance property of [11] is much better than that of [8], if v is big enough.

3.4 The construction based on the pseudo-symplectic space

In 2010, Li et al. [18] constructed two family pooling designs on pseudo-symplectic spaces $\mathbb{F}_q^{(2\nu+1)}$.

Definition 3.22 [18] For $2r + 1 \leq d_0 < d < k \leq \nu + r + 1$, assume that P_0 is a fixed subspace of type $(d_0, 2r + 1, r, 0)$ in $\mathbb{F}_q^{(2\nu+1)}$. Let M be a binary matrix whose columns (rows) are indexed by all subspaces of type $(k, 2r + 1, r, 0)$ containing P_0 (subspaces of type $(d, 2r + 1, r, 0)$ containing P_0) in $\mathbb{F}_q^{(2\nu+1)}$ such that $M(A, B) = 1$ if $A \subseteq B$, and 0 otherwise. This matrix is denoted by $M_1(n, d, k)$.

Theorem 3.23 [18] *Suppose* $2r + 1 \leq d_0 < d < k \leq \nu + r + 1$ *and set* $b = \frac{q(q^{k-d_0-1}-1)}{q^{k-d}-1}$. *Then* $M_1(n, d, k)$ *is* s^e*-disjunct for* $1 \leq s \leq b$ *and*

$$
\begin{aligned}
e = q^{k-d} & N(d-d_0-1, 0, 0, 0; k-d_0-1, 0, 0, 0; 2(\nu-r+d_0)) \\
& -(s-1)q^{k-d-1}N(d-d_0-1, 0, 0, 0; k-d_0-2, 0, 0, 0; 2(\nu-r+d_0)),
\end{aligned}
$$

where $N(m_1, 0, 0, 0; m, 0, 0, 0; 2\nu+\delta)$ *is from* (5).

Definition 3.24 For $3 \leq 2r+1 \leq d < k \leq \nu+r+1$, let M be a binary matrix whose columns (rows) are indexed by all subspaces of type $(k, 2r+1, r, 1)$ $((d, 2r+1, r, 1))$ in $\mathbb{F}_q^{(2\nu+1)}$ such that $M(A, B) = 1$ if $A \subseteq B$, and 0 otherwise. This matrix is denoted by $M_2(n, d, k)$.

Theorem 3.25 [18] *Suppose* $3 \le 2r+1 \le d-1 < k-2 \le \nu+r+1$. *If* $1 \le s \le q^{2r}$, *then* $M_2(n, d, k)$ *is* s^e*-disjunct, where* $e = q^{(k-d-1)(d-1)+2r}$.

The test efficiency of construction is smaller than that of [2] under some conditions.

3.5 The constructions based on the unitary space

3.5.1 Use subspaces containing a fixed $(m_0, 0)$*-space and contained in its dual space*

In 2009, Zhang et al. [34] constructed a family pooling designs based on the unitary space.

Definition 3.26 [34] Select integers $0 \le m_0 < r < m \le \nu$. Assume P_0 is a fixed $(m_0, 0)$-space of $\mathbb{F}_{q^2}^{(n)}$. Let M be the (0, 1)-matrix by taking all $(m, 0)$-spaces which are contained in $P_0^{\perp}$ and contain P_0 as columns and all $(r, 0)$-spaces which are contained in $P_0^{\perp}$ and contain P_0 as rows. M has a 1 in row i and column j if and only if i is contained in j .

Theorem 3.27 [34] *Suppose* $m-r \ge 2$ *and set* $b = \frac{(q^{2(m-m_0-1)}-1)(q^{2(m-m_0)}-q^{2(m-r-2)})}{(q^{2(m-r-2)}-1)(q^{2(m-m_0-1)}-1)-1}$. *Then* M *is* d^z*-disjunct for* $1 \le d \le b$ *and*

$$z = \begin{bmatrix} m-m_0 \\ r-m_0 \end{bmatrix}_{q^2} \begin{bmatrix} m-m_0-1 \\ r-m_0 \end{bmatrix}_{q^2} + (d-1) \begin{bmatrix} m-m_0-2 \\ r-m_0 \end{bmatrix}_{q^2}.$$

The test efficiency of construction is smaller than that of [2] under some conditions.

3.5.2 Use subspaces containing a fixed (d_0, r)*-space*

In 2011, Gao et al. [10] presented two new constructions based on the unitary space.

Definition 3.28 [10] For $2r - 2d_0 < 2d < 2k \le n + r$, assume that P_0 is a fixed subspace of type (d_0, r) in $\mathbb{F}_{q^2}^{(n)}$. Let M be a binary matrix whose columns (rows) are indexed by all subspaces of type (k, r) containing P_0 (subspaces of type (d, r) containing P_0) in $\mathbb{F}_{q^2}^{(n)}$ such that $M(A, B) = 1$ if $A \subseteq B$ and 0 otherwise. This matrix is denoted by $M_1(n, d, k)$.

Theorem 3.29 [10] *Suppose* $2r \le 2d_0 < 2d < 2k \le n + r, r = 2s + \delta_1$, *where* $\delta_1 = 0, 1$ *and set* $b = \frac{q^2(q^{2(k-d_0-1)}-1)}{q^{2(k-d)}-1}$. *Then* $M_1(n, d, k)$ *is* l^e*-disjunct for* $1 \le l \le b$ *and*

$$\begin{aligned} e = {} & q^{2(k-d)} N(d-d_0-1, 0; k-d_0-1, 0; 2(\nu+s-d_0)) \\ & -(l-1)q^{2(k-d-1)} N(d-d_0-1, 0; k-d_0-2, 0; 2(\nu+s-d_0)), \end{aligned}$$

where $N(m_1, r; m, r; n)$ is from (6).

Definition 3.30 [10] For $2 \le 2r \le 2d < 2k \le n+r$, let M be a binary matrix whose columns (rows) are indexed by all subspaces of type (k, r) $((d, r))$ in $\mathbb{F}_{q^2}^{(n)}$ such that $M(A, B) = 1$ if $A \subseteq B$ and 0 otherwise. This matrix is denoted by $M_2(n, d, k)$.

Theorem 3.31 [10] *Suppose* $0 \le 2r-4 \le 2d < 2k-2 \le n+r-2$. *If* $1 \le s \le q^{2r}$, *then* $M_2(n, d, k)$ *is* s^e*-disjunct, where* $e = q^{2(k-d-1)d+2r}$.

The test efficiency of construction is smaller than that of [2] under some conditions.

3.5.3 Use totally isotropic subspaces

In 2010, Guo [4] constructed a family of inclusion matrices associated with subspaces of $\mathbb{F}_{q^2}^{(2\nu+\delta)}$, and exhibited its disjunct property.

Definition 3.32 [4] In the unitary space $\mathbb{F}_{q^2}^{(2\nu+\delta)}$, given integers $1 \le r \le \lfloor \frac{m}{2} \rfloor$ and $m < 2\nu+\delta$. Let $M(r, m; 2\nu+\delta)$ be the binary matrix whose rows (resp. columns) are indexed by $\mathcal{M}(r, 0; 2\nu+\delta)$ (resp. $\mathcal{M}(m, m; 2\nu+\delta)$). We also order elements of these sets lexicographically. $M(r, m; 2\nu+\delta)$ has a 1 in row i and column j if and only if the i-th subspace of $\mathcal{M}(r, 0; 2\nu+\delta)$ is a subspace of the j -th subspace of $\mathcal{M}(m, m; 2\nu+\delta)$.

Theorem 3.33 [4] *Let* $1 \le r \le \lfloor \frac{m}{2} \rfloor$ *and* $m < 2\nu+\delta$, *and let* $\beta = N(r, 0; m, m; 2\nu+\delta)$, $\gamma = N(r, 0; m-1, m-1; 2\nu+\delta)$, $\xi = N(r, 0; m-1, m-2; 2\nu+\delta)$, $\eta = N(r, 0; m-2, m-2; 2\nu+\delta)$, $\zeta = N(r, 0; m-2, m-3; 2\nu+\delta)$, $\rho = N(r, 0; m-2, m-4; 2\nu+\delta)$, $\alpha = \max\{\gamma-\eta, \gamma-\zeta, \gamma-\rho, \xi-\eta, \xi-\zeta, \xi-\rho\}$, *where* $N(m_1, r; m, r; n)$ *is from* (6). *Then the following (i)–(iii) hold:*

(i) *For* $m \ge 4$. *If* $1 \le d \le \left\lfloor \frac{\beta-\max\{\gamma,\xi\}-1}{\alpha} \right\rfloor + 1$, *then* $M(r, m; 2\nu+\delta)$ *is* d^e*-disjunct, where* $e = \beta - \max\{\gamma, \xi\} - (d-1)\alpha - 1$. *In particular, if* $1 \le d \le \min\{\left\lfloor \frac{\beta-\max\{\gamma,\xi\}-1}{\alpha} \right\rfloor + 1, q^2+1\}$,*then exist* $d+1$ *distinct columns of* $M(r, m; 2\nu+\delta)$, *i.e.,* $d+1$ *distinct* m*-dimensional non-isotropic subspaces of* $\mathbb{F}_{q^2}^{(2\nu+\delta)}$, *such that the* $d+1$ *subspaces contain same* $(m-2)$*-dimensional subspace* P *and the number of* r*-dimensional totally isotropic subspaces contained in* P *is equal to* $\min\{\eta, \zeta, \rho\}$.

(ii) *For* $m = 3$. *If* $1 \le d \le q^2 - q$, *then* $M(r, m; 2\nu+\delta)$ *is fully* d^e*-disjunct, where* $e = q^3 - d(q+1)$.

(iii) *For* $m = 2$. *If* $1 \le d \le q$, *then* $M(1, 2; 2\nu+\delta)$ *is fully* d^e*-disjunct, where* $e = q - d$.

The test efficiency of construction is smaller than that of [2] under some conditions.

3.6 The constructions based on the orthogonal space

3.6.1 Use subspaces containing a fixed $(m_0, 0, 0)$*-space and contained in its dual space*

In 2009, Zhang et al. [35] constructed a family pooling designs.

Definition 3.34 [35] Select integers $0 \leq m_0 < r < m \leq \nu$. Assume P_0 is a fixed $(m_0, 0, 0)$-subspace of $\mathbb{F}_q^{(2\nu+\delta)}$. Let M be the (0, 1)-matrix, where the columns (rows) are labeled by $(m, 0, 0)$-subspaces ($(r, 0, 0)$-subspaces), which are contained in $P_0^{\perp}$ and contain P_0. $m_{ij} = 1$ if and only if the label of row i is contained in the label of column j.

Theorem 3.35 [35] *Suppose* $m - r \geq 2$ *and set* $b_1 = \frac{q(q^{m-m_0-1}-1)}{q^{m-r}-1}$. *Then* M *is* d^z*-disjunct when* $1 \leq d \leq b_1$, *where*

$$\begin{aligned} z = &N(r - m_0, 0, 0; m - m_0, 0, 0; 2(\nu - m_0) + \delta, \Delta) \\ &- dN(r - m_0, 0, 0; m - m_0 - 1, 0, 0; 2(\nu - m_0) + \delta, \Delta) \\ &+ (d - 1)N(r - m_0, 0, 0; m - m_0 - 2, 0, 0; 2(\nu - m_0) + \delta, \Delta) \end{aligned}$$

where $N(m_1, 0, 0; m, 0, 0; 2\nu + \delta, \Delta)$ *is from* (7).

The test efficiency of construction is smaller than that of [2] under some conditions.

3.6.2 Use subspaces containing a fixed $(d_0, 2r + 1, r, 1)$*-space*

In 2010, Li et al. [17] constructed two family pooling designs.

Definition 3.36 [17] For $2r + 1 \leq d_0 < d < k \leq \nu + r + 1$, assume that P_0 is a fixed subspace of type $(d_0, 2r + 1, r, 1)$ in $\mathbb{F}_q^{(2\nu+1)}$. Let M be a binary matrix whose columns (rows) are indexed by all subspaces of type $(k, 2r+1, r, 1)$ containing P_0(all subspaces of type $(d, 2r + 1, r, 1)$ containing P_0) in $\mathbb{F}_q^{(2\nu+1)}$ such that $M(A, B) = 1$ if $A \subseteq B$ and 0 otherwise. This matrix is denoted by $M_1(\nu, d, k)$.

Theorem 3.37 [17] *Suppose* $2r + 1 \leq d_0 < d < k \leq \nu + r + 1$ *and set* $b = \frac{q(q^{k-d_0-1}-1)}{q^{k-d}-1}$. *Then* $M_1(\nu, d, k)$ *is* s^e*-disjunct for* $1 \leq s \leq b$ *and*

$$\begin{aligned} e = &q^{k-d} N(d - d_0 - 1, 0, 0; k - d_0 - 1, 0, 0; 2(\nu + r + 1 - d_0)) \\ &-(s - 1)q^{k-d-1} N(d - d_0 - 1, 0, 0; k - d_0 - 2, 0, 0; 2(\nu + r + 1 - d_0)), \end{aligned}$$

where $N(m_1, 0, 0; m, 0, 0; 2\nu)$ *is from* (8).

The test efficiency of construction is smaller than that of [2] under some conditions.

Definition 3.38 For $3 \leq 2r+1 \leq d < k \leq \nu+r+1$, let M be a binary matrix whose columns (rows) are indexed by all subspaces of type $(k, 2r + 1, r, 1)$(all subspaces of type $(d, 2r + 1, r, 1)$) in $\mathbb{F}_q^{(2\nu+1)}$ such that $M(A, B) = 1$ if $A \subseteq B$ and 0 otherwise. This matrix is denoted by $M_2(\nu, d, k)$.

Theorem 3.39 [17] *Suppose* $3 \leq 2r+1 \leq d-1 < k-2 \leq \nu+r-1$. *If* $1 \leq s \leq q^{2r}$, *then* $M_2(\nu, d, k)$ *is* s^e*-disjunct, where* $e = q^{(k-d-1)(d-1)+2r}$.

The test efficiency of construction is smaller than that of [2] under some conditions.

3.7 The constructions based on the affine-symplectic space

In 2011, Gao et al. [9] constructed two family pooling designs based on the affine-symplectic space.

Definition 3.40 [9] For $2r - d_0 < d < k \leq \nu + r$, assume that $y_0 + P_0$ is a fixed (d_0, r)-flat of $\mathrm{ASG}(2\nu, \mathbb{F}_q)$. Let M be a binary matrix whose columns (rows) indexed by all (k, r)-flats containing $y_0 + P_0((d, r)$-flats containing $y_0 + P_0)$ in $\mathrm{ASG}(2\nu, \mathbb{F}_q$) such that $M(x + A, y + B) = 1$ if $x + A \subseteq y + B$ and 0 otherwise. This matrix is denoted by $M_1(\nu, d, k)$.

Theorem 3.41 [9] *Suppose* $2r \leq d_0 < d < k \leq \nu + r$ *and set* $b = \frac{q(q^{k-d_0-1}-1)}{q^{k-d}-1}$. *Then* $M_1(\nu, d, k)$ *is* s^e*-disjunct for* $1 \leq s \leq b$ *and*

$$e = q^{k-d} N(d - d_0 - 1, 0; k - d_0 - 1, 0; 2(\nu + r - d_0)) \\ -(s - 1)q^{k-d-1} N(d - d_0 - 1, 0; k - d_0 - 2, 0; 2(\nu + r - d_0)),$$

where $N(m_1, r; m, r; 2\nu)$ *is from* (4).

The test efficiency of construction is smaller than that of [2] under some conditions.

Definition 3.42 For $2 - 2r - d < k \leq \nu + r$, let M be a binary matrix whose columns (rows) indexed by all (k, r)-flats ((d, r)-flats) in $\mathrm{ASG}(2\nu, \mathbb{F}_q)$ such that $M(x + A, y + B) = 1$ if $x + A \subseteq y + B$ and 0 otherwise. This matrix is denoted by $M_2(\nu, d, k)$.

Theorem 3.43 [9] *Suppose* $4 \leq 2r + 2 \leq d < k - 1 \leq \nu + r - 1$. *If* $1 \leq s \leq q^{2r+1}$, *then* $M_2(\nu, d, k)$ *is* s^e*-disjunct, where* $e = q^{(k-d-1)(d+1)+2r+1}$.

The test efficiency of construction is smaller than that of [2] under some conditions.

4 Conclusion

The algebraic construction is motivated from containment design which was initiated by Macula [21,22]. Ngo and Du [26,27] generalized containment design to a more general setting. Park et al. [28] employed simplicial complex to perform the idea of containment design. Huang and Weng [12] gave a very general theorem for containment design and suggested to use linear spaces as tools, which initiated the algebraic construction. Note that geometric lattices are among pooling spaces. In [13] the authors gave some examples of pooling spaces from geometry of classical groups over finite fields. So, in this sense the construction based on geometry of classical groups over finite fields was initialed by [13]. This construction opened a new door for studying pooling designs. Current results in literature (Dyachkov et al. [2]; Lang et al. [15]) have showed that many spaces can be involved in algebraic construction. Therefore, the idea is powerful. It may be possible to extend to pooling design of other types with other applications (Lang et al. [15]; Macula et al. [24]).

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