VFAST Transactions on Mathematics
http://vfast.org/journals/index.php/VTM@ 2021, ISSN(e): 2309-0022, ISSN(p): 2411-6343
Volume 9, Number 1, January-December, 2021 pp: 28-36

Structures of Digraphs Arizing from Lambert Type Maps

Tayyiba Sabahat*, **Sufyan Asif**[1], **Asif Abd ur Rehman**[1]

[1] Department of Mathematics, University of the Punjab, Lahore Pakistan

Subject Classification:
05C25, 11E04, 20G15

Journal Info:
Submitted:
July 15, 2021
Accepted:
October 10, 2021
Published:
December 31, 2021

Abstract The Well-known function We^W is called Lambert Map. This map has been viewed by many researchers for finding the approximate solutions of exponential function especially in numerical analysis. Later, it has been incorporated in number theory for finding integral solutions of exponential congruences under a fixed modulus. Instead of We^W, we use the function $W2e^W$ and call this function as Discrete Lambert Type Function (DLTF). In this work, we produce graphs using DLTF, and discuss their structures. We show that the digraphs over DLTF satisfy many structures as these have been followed for using We^W. It would be of great interest, if these results could be generalized for We^W for all integers n, in the future.

Keywords: Digraph, Lambert Type Map, Residues

***Correspondence Author Email Address:**
sabahattayyiba@gmail.com

1 Introduction

To find the solution of equations in which unknown also appears in exponential terms The Lambert W functions are used. It is defined as $z = W(z)e^{W(z)}$, where z is a complex number. Equivalently it can be defined as $f(x) = xe^x$. In 1758 a Diophantine equation $y = y^k + t$ was solved by Lambert[1].In 1980 Lambert function was stored in MCAS(Maple Computer Algebraic System) as a function to find out the solution of the algebraic equations in which exponential or logarithmic functions are involved[2]. In 1993 it was revealed that the Lambert function gives an exact solution to the quantum-mechanical double-well Dirac delta function model for equal charges[3]. In 2016 Khalid and Lubna proposed loops in digraphs of Lambert Mapping modulo of prime powers and its applications[4].

Definition 1. *Graph A graph is an ordered pair $G(V, E)$ which consists on two sets V and E, where V is set of points named as set of vertices and E is a set of edges.*

Definition 2. *Digraph A graph in which edges have direction is called directed graphs or digraphs.*

2 Special Images

: A number β is said to be fixed point of DLTM iff $\beta^2 g^\beta \equiv \beta (mod m)$ for a positive integer m. The following results are elaborating fixed points and image structures for specific numbers.

Theorem 1. *Let f be a DLTM and s be any odd prime. If s is any odd prime, and h = 2 then t = s − 2 always map on 2 (mod s).*

Proof. Suppose

$$\begin{aligned} f(t) &= t^2 h^t \\ \text{then,} \quad f(s-2) &\equiv (s-2)^2 (2)^{s-2} \ (mod\ s) \\ &\equiv (s-2)^2 2^s 2^{-2} \ (mod\ s) \end{aligned}$$

Using Euler's formula, we have

$$\begin{aligned} f(s-2) &\equiv ((s-2)^2 2^{-1}) \ (mod\ s) \\ &\equiv ((-2)^2 2^{-1}) \ (mod\ s) \\ &\equiv 2 \ (mod\ s). \end{aligned}$$

□

Proposition 1. *Let f be a DLTM and h = s − 1. If t is odd then f(t) = s − t^2 and if t is even then f(t) $\equiv t^2$ (mod s).*

Proof. Let t be any odd integer. then

$$t = 2k + 1 \tag{1}$$

As $h = s - 1$, w see that

$$f(2k+1) = (2k+1)^2 (s-1)2k + 1,$$

Now since $(s-1)^2 \equiv 1 \ (mod\ s)$

$$\begin{aligned} &\equiv ((2k+1)^2 (s-1)^{2k} (s-1)) \ (mod\ s) \\ &\equiv ((2k+1)^2 (s-1)) \ (mod\ s) \\ &\equiv ((4k^2 + 1 + 4k)(s-1)) \ (mod\ s) \\ &\equiv -(2k+1)^2 \\ &\equiv -t^2 \\ &\equiv s - t^2 \ (mod\ s), \end{aligned}$$

Also, if t is even then it is of the form

$$\begin{aligned} t &= 2k \\ f(t) &= t^2 h^t \\ &\equiv (2k)^2 (s-1)^{2k} \ (mod\ s) \\ &\equiv (2k)^2 \\ &\equiv t^2 \ (mod\ s). \end{aligned}$$

□

3 Power Residue

Recall that an integer α is a power pth residue of any prime s if the congruence $t^p \equiv \alpha$ admits a solution in t modulo s. In the following result, we discuss the power residues of the map DLTM.

Theorem 2. *Let h be an nth power residue of s. Then, t^2 is an nth power residue of s if and only if t^2h^t is an nth power residue of s.*

Proof. Assume first that t^2 is an nth power residue of s then there must exist some integer b such that it satisdies

$$y^n \equiv s^2 \pmod s, \tag{2}$$

Equivalently.

$$b^n \equiv t^2 \pmod s \tag{3}$$

now since h is an nth power residue of s so we will find some integer for u such that

$$u^n \equiv h \pmod s, \tag{4}$$

This means that

$$\begin{aligned} t^2h^t &\equiv b^n(u^n)^t & (5)\\ &\equiv (bu^t)^n. & (6)\end{aligned}$$

this clearly shows that t^2h^t is an nth power residue of s
Conversely, suppose t^2h^t ($mod\ s$) is an nth power residue. Then we can write

$$t^2h^t \equiv z^n \quad mod\ s \qquad for\ \ some\ \ value\ \ of\ \ z\epsilon(Z/SZ)^*, \tag{7}$$

h is an nth power residue

$$h = y^n \quad mod\ s\ \ for\ \ some\ \ y\epsilon(Z/SZ)^*, \tag{8}$$

$$\begin{aligned} t^2h^t &\equiv z^n & (9)\\ t^2h^th^{-t} &\equiv z^nh^{-t} & (10)\\ &\equiv z^n(y^n)-t & (11)\\ t^2 &\equiv z^n(y^{-t})^n & (12)\\ &\equiv (zy^{-t})^n \pmod s, & (13)\end{aligned}$$

t^2 is an nth power residue. □

The following result is easy to prove.

Corollary 1. *If h is quadratic residue then $\frac{s-1}{2} \rightarrow (\frac{s-1}{2})^2 \pmod s$.*

Theorem 3. *If s is quadratic residue then $s - 1 \rightarrow 1$.*

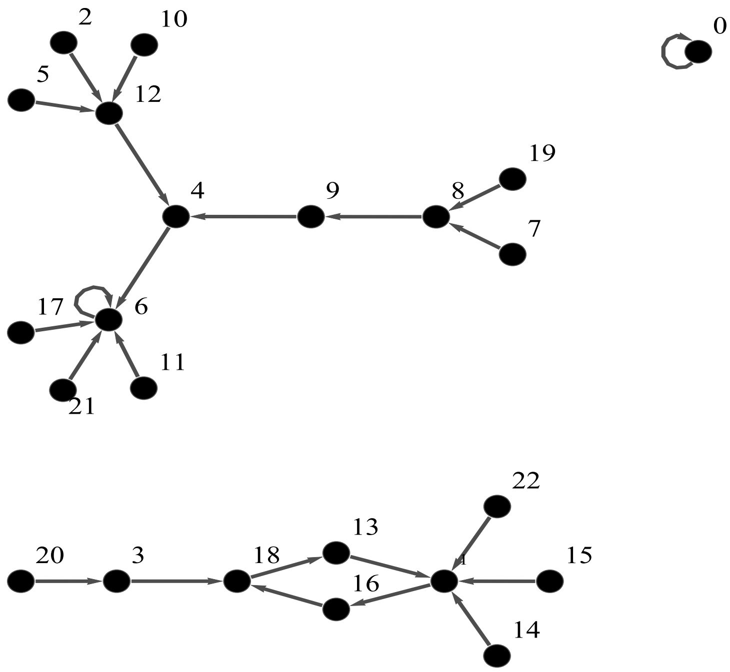

Figure 1. Shows the diagraph $G(16, 23)$

Proof. Let $f(t) = t^2h^t$ be a DLTM, then

$$f(s-1) = (s-1)^2h^{s-1}, \tag{14}$$

$As a^{s-1} \equiv 1 \pmod s$
$\therefore$, $h^{s-1} \equiv 1 \pmod s$, an hence, $(s-1)^2 \equiv 1 \pmod s$.
Or $f(s-1) \equiv 1 \pmod s$. □

4 Continuity of the Map

In this section, we discuss uniformly continuity and continuity of the map for references.

Theorem 4. *If q is an odd prime $d \in Z_q$ is fixed, and $h \in 1 + qZ_q$, then $f(t) = t^2h^t - d$ is uniformly continuous for $t \in Z$.*

Proof. Suppose $h = 1 + qA$ where $A \in Z_q$. We're aware of that for any $\varepsilon > 0$, there exist an N such that $q^{-N} < \varepsilon$.
Let $l, m \in Z$ s.t

$$|l - m|_q \leq q^{-N} < q^{-(N-1)} \quad = \quad \delta \tag{15}$$

or $(l - m) \in q^N Z_q$, and $l = m + bq^N$ where $b \in Z$, then we must demonstrate that

$$|l^2h^l - d - (m^2h^m - d)|_q \quad < \quad \varepsilon \tag{16}$$

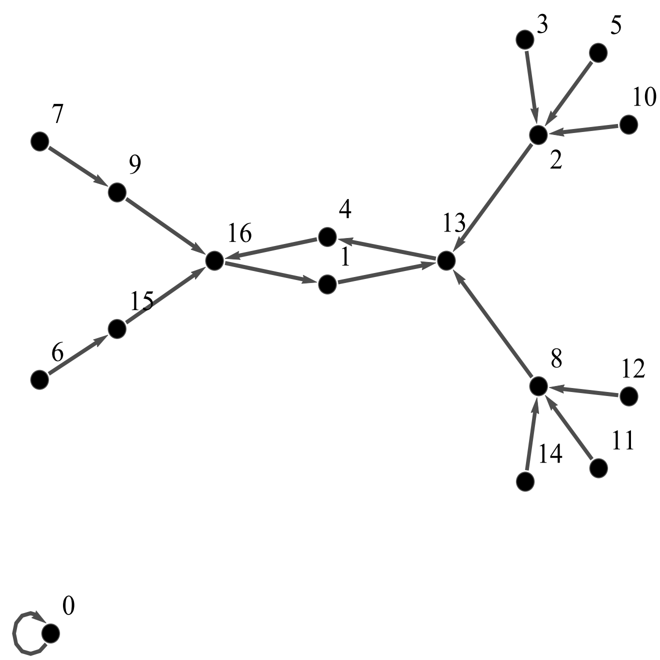

Figure 2. Shows the diagraph $G(13, 17)$

Note that

$$h^{bq^N} = (1 + qA)^{bq^N} \quad (17)$$
$$= 1 + bq^N qA + \ldots + (qA)^{bq^N} \quad (18)$$
$$\in 1 + q^N Z_q. \quad (19)$$

So we know that $h^{bq^N} - 1 \in q^N Z_q$, or $|h^{bq^N} - 1|_q \leq q^{-N}$. Moreover, since $m \in Z$, $|m|_q \leq 1$. Notice that $|bq^N h^{bq^N}|_q \leq q^{-N}$. Now, consider

$$|l^2 h^l - m^2 h^m| = |(m + bq^N)^2 h^{m+bq^N} - m^2 h^m|_q \quad (20)$$
$$= |(m^2 + b^2 q^{2N} + 2mbq^N) h^{m+bq^N} - m^2 h^m|_q \quad (21)$$
$$= |m^2 h^{m+bq^N} + b^2 q^{2N} h^{m+bq^N} + 2mbq^N h^{m+bq^N} - m^2 h^m|_q \quad (22)$$
$$= |h^m|_q |m^2 g^{bq^N} + b^2 q^{2N} h^{bq^N} + 2mbq^N h^{bq^N} - m^2|_q. \quad (23)$$

Since $h \in 1 + qZ_q$, $|h^m|_q = 1$,

$$= |m^2 h^{bq^N} + b^2 q^{2N} h^{bq^N} + 2mbq^N h^{bq^N} - m^2|_q \quad (24)$$
$$= |m^2 (h^{bq^N} - 1) + bq^N h^{bq^N} (bq^N + 2m)|_q, \quad (25)$$

$\because \alpha = bq^N + 2m$,

$$\leq \max(|h^{bq^N} - 1||h|_q, |\alpha bq^N h^{bq^N}|_q) \quad (26)$$
$$\leq q^N. \quad (27)$$

□

5 Special solutions of the Map

Its a converse back way to write solutions of the DLTM type congruences modulo prime number using digraphs. As we are interested to tell after observing the digraphs that which type of congruences are satisfying some special solution and whats such solutions count. The following theorem is an attempt to incorporate the idea behind.

Theorem 5. *Let r be an odd prime. For a fixed $m \in \{1, \ldots, r^e\}$ and for $r \nmid h$ and $d \in \{1 \ldots r^{e-1}(r-1)\}$, if we consider $m^2h^m \equiv d \pmod{r^e}$ and let $m^2(h^{-1})^m \equiv \acute{d} \pmod{r^e}$, then $d.\acute{d} \equiv m^4 \pmod{r^e}$. Moreover, if we let $m^2(-h)^m \equiv \dot{d} \pmod{r^e}$ then $\dot{d} \equiv (-1)^m d \pmod{r^e}$.*

Proof. We show that $d.\acute{d} \equiv t^4 \pmod{r^e}$. Since $m^2h^m \equiv d \pmod{r^e}$ and $m^2(h^{-1})^m \equiv \acute{d} \pmod{r^e}$, We could say that

$$\begin{aligned} d.\acute{d} &\equiv (m^2h^m)(m^2(h^{-1})^m) \pmod{r^e} && (28)\\ &\equiv m^4(h^m)(h^{-m}) \pmod{r^e} && (29)\\ &\equiv m^4 \pmod{r^e}. && (30) \end{aligned}$$

Hence, $d.\acute{d} \equiv t^4 \pmod{r^e}$. Now we must demonstrate that $\dot{d} \equiv (-1)^m d \pmod{r^e}$,

$$\begin{aligned} \dot{d} &\equiv m^2(-h)^m \pmod{r^e} && (31)\\ &\equiv m^2(-1)^m h^m \pmod{r^e} && (32)\\ &\equiv (-1)^m m^2 h^m \pmod{r^e} && (33)\\ &\equiv (-1)^m d \pmod{r^e}. && (34) \end{aligned}$$

Thus

$$\dot{d} \equiv (-1)^m d \pmod{r^e}.$$

□

Proposition 2. *Let r be an odd prime and h be a generator modulo r^e. If $d = \frac{r^e + re + 1}{4}$ and $r^e - 1$ is the only element of order 2 in the multiplicative group of Z/r^eZ, Then $m = \frac{r^e - r^{e-1}}{2}$ is one of the solution to*

$$m^2h^m \equiv d \pmod{r^e}. \quad (35)$$

Proof. We can see that by hypothesis

$$\begin{aligned}
m^2h^m - d &= (\frac{r^e - r^{e-1}}{2})^2 h^{\frac{r^e - r^{e-1}}{2}} - \frac{r^e + r^{e+1}}{4} && (36)\\
&\equiv (\frac{r^e - r^{e-1}}{2})^2 (r^e - 1) - \frac{r^e + r^{e+1}}{4} \ (mod\ r^e) && (37)\\
&= \frac{(r^{2e} - r^{2e-2} - 2r^{e+e-1})(r^e - 1)}{4} - \frac{r^e + r^{e+1}}{4} \ (mod\ r^e) && (38)\\
&= \frac{r^e(r^e - r^{e-2} - 2r^{e-1})(r^e - 1)}{4} - \frac{r^e + r^{e+1}}{4} \ (mod\ r^e) && (39)\\
&= \frac{r^e(r^e - r^{e-2} - 2r^{e-1})(r^e - 1) - r^e - r^{e+1}}{4} \ (mod\ r^e) && (40)\\
&= \frac{r^e(r^e - r^{e-2} - 2r^{e-1})(r^e - 1) - r^e(1 + r)}{4} \ (mod\ r^e) && (41)\\
&= \frac{r^e[(r^e - r^{e-2} - 2r^{e-1})(r^e - 1) - (1 + r)]}{4} \ (mod\ r^e) && (42)\\
&\equiv 0 \ (mod\ r^e). && (43)
\end{aligned}$$

□

6 Cycles

Recall that the vertices $v_1, v_2, ..., v_{k-1}, v_k$ forms a k length cycle if and only if

$$\begin{aligned}
{v_1}^2 h^{v_1} &\equiv v_2 \ (\ mod\ m)\\
{v_2}^2 h^{v_2} &\equiv v_3 \ (\ mod\ m)\\
&\vdots\\
{v_k}^2 h^{v_k} &\equiv r_1 \ (\ mod\ m),
\end{aligned}$$

The following theorem illustrate the cycic behaviour of the map DLTM, Let $f^n(t)$ denote the function $f(t) = t^2h^t$ mod s applied $n+1$ times (e.g$f^2(t) = f(f(t))$). Then

$$\begin{aligned}
f^n(t) &= (f^{n-1}(t))^2 h^{f^{n-1}(t)} && (44)\\
f^{n+1}(t) &= (f^n(t))^2 h^{f^n(t)}. && (45)
\end{aligned}$$

Proof. we prove it by induction.
Let

$$\begin{aligned}
f^0(t) &= t && (46)\\
f^1(t) &= f(t) = t^2 h^t, && (47)
\end{aligned}$$

suppose $n = k$

$$(f^n(t)) = (f^{n-1}(t))^2 h^{f^{n-1}(t)} \ (mod\ s). \quad (48)$$

Then $n = k + 1$

$$
\begin{aligned}
f(f^{n}(t)) &= f((f^{n-1}(t))^{2}h^{f^{n-1}(t)}) && (49)\\
&= ((f^{n-1}(t))^{2}h^{f^{n-1}(t)})^{2}h^{(f^{n-1}(t))^{2}h^{f^{n-1}(t)}} && (50)\\
&= (f^{n-1}(t))^{4}h^{2f^{n-1}(t)}(f^{n-1}(t))^{2}h^{f^{n-1}(t)} && (51)\\
&= (f^{n-1}(t))^{2}h^{f^{n-1}(t)}(f^{n-1}(t))^{2}h^{f^{n-1}(t)}(f^{n-1}(t))^{2}h^{f^{n-1}(t)} && (52)\\
&= f(n)f(n)h^{f^{n}(t)} && (53)\\
f^{n+1}(t) &= (f(n))^{2}h^{f^{n}(t)}. && (54)
\end{aligned}
$$

□

The following are the few results without proof regarding cycles and components

Theorem 6. *The Functional graphs of $t \mapsto t^2h^t \pmod s$, $h = s-1$ and $s \equiv 3 \pmod 4$ are the union of disjoint cycle, here s is prime.*

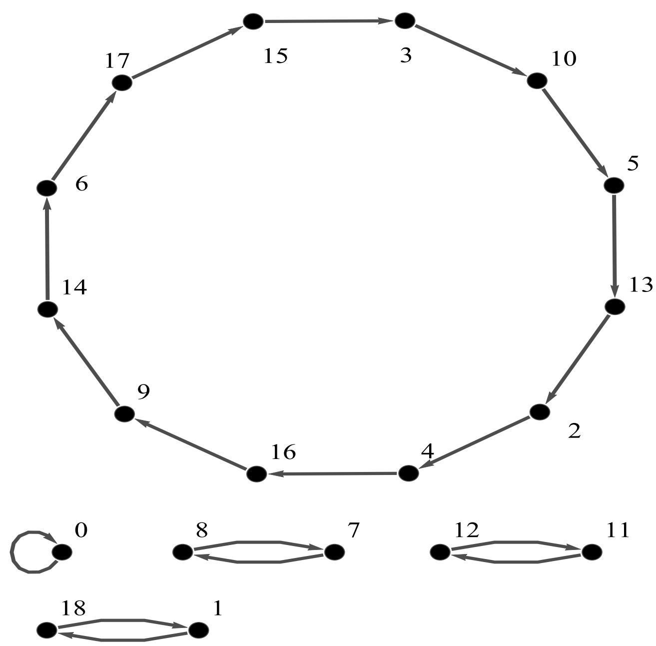

Figure 3. Shows the diagraph $G(18, 19)$

Theorem 7. *Let f be DLTM. Let s be an odd prime. if $h = s - 1$ there exist a cycle of length two between* 1 *and* $s - 1$.

Theorem 8. *Every component contains a cycle.*

Theorem 9. *Let f be DLTM. There exists a cycle of length $w > 1$ in $G(n)$ under DLTM, $f(t) = t^2h^t$ with $ord_n^h \mid t$ if and only if w is the least positive integer such that $2^w \equiv 1 \pmod s$, where s is odd positive divisor of $\lambda(n)$, and $\lambda(n)$ is the Charmical Lambda*

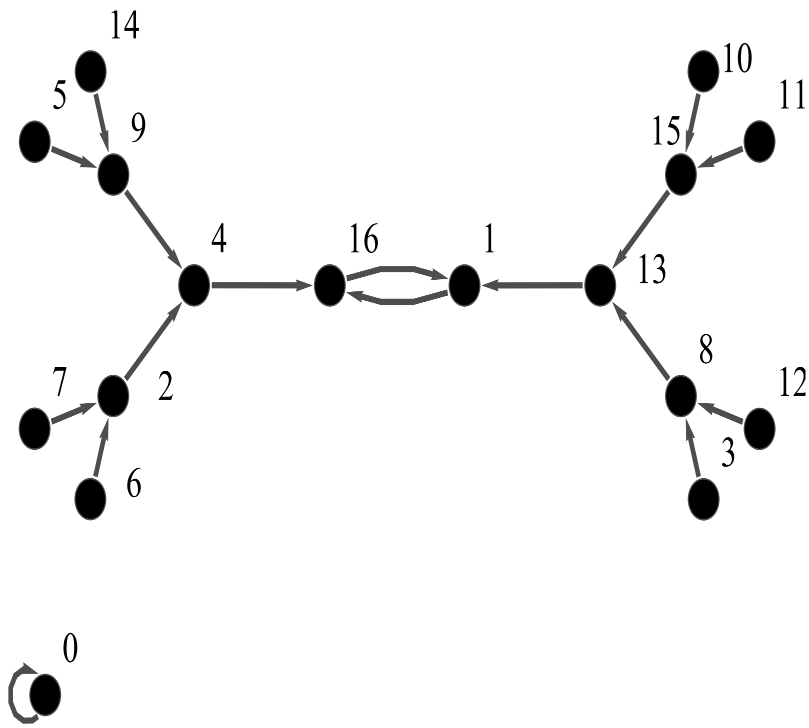

Figure 4. Shows the diagraph $G(16, 17)$

References

[1] *Lambert, J.H. (1758) Observationes Variae in Mathesin Puram. Acta Helvetica Physico-Mathematico-Anatomico-Botanico- Medica, 3, 128-168.*

[2] *Corless, R.M., Gonnet, G.H., Hare, D.E.G. and Jeffrey, D.J. (1993) Lamberts W Function in Maple. The Maple Technical Newsletter (MapleTech), 9, 12-22.*

[3] *Scott, TC; Babb, JF; Dalgarno, A; Morgan, John D (Aug 15, 1993). "The calculation of exchange forces: General results and specific models". J. Chem. Phys. American Institute of Physics. 99 (4): 28412854.*

[4] *M. Khalid Mahmood, Lubna Anwar(2016).Loops in Digraphs of Lambert Mapping Modulo Prime Powers: Enumerations and Applications.Advances in Pure Mathematics, 6, 564-570.*