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Solutions of negative Pell's equation involving proth prime

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Abstract

Many researchers have been devoted to finding the solutions (η, ζ) in the set of nonnegative integers, of Diophantine equation (Pell Equation) of the type $\eta^2 = D\zeta^2 \pm \alpha$, where the value α is fixed positive integers. In this article, we look for non-trivial integer solutions to two negative Pell equations $x^2 = 13y^2 - 17^t$, and $x^2 = 113y^2 - 97^t, t \in N$ for the different choices of t particular by

(i) $t = 1$

(ii) $t = 3$,

(iii) $t = 5$,

(iv) $t = 2k$,

(v) $t = 2k + 5$, $\forall k \in N$.

Additionally, recurrence relations on the solutions are obtained.

Keywords

Diophantine Equation, Integral solution, Pell's Equation, Brahma Gupta Lemma, Proth Prime.

AMS Subject Classification

11D09, 11D61, 11D72.

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Contents

1. Introduction

Number theory is the branch of Mathematics concerned with studying the properties and relations of integers. Many of these problems are concerned with the properties of prime numbers. Number theory includes the different aspects of natural numbers and their extensions in various fields of Mathematics and Science. There are number of branches of number theory including algebraic number theory, analytic number theory, geometric number theory, combinatorial number theory, computational number theory, probabilistic number theory and so on. Number theory also includes the study of irrational numbers, transcendental numbers, continued fractions and Diophantine equation [4,5,6].

Diophantine equations are named after the Greek Mathematician Diophantus of Alexandria, whose book Arithmetica included a study of such equations $[13-16]$.A Diophantine equation is a numerical polynomial equation that has more than one unknown quantity and integer coefficients, and for which a solution is sought in integers [$1-5$].For example $23x + 21y + 7 = 0$ is a Diophantine equation, where x and y are the unknown quantities. Famous examples of Diophantine equations include Pythagoras' theorem and Fermat's last theorem. Individual Diophantine problems were studied by such great Mathematicians like Euler, Gauss and Fermat.

Pierre de Fermat was a 17 th century French lawyer and amateur Mathematician. He is often credited with founding modern number theory and he made some of the greatest advances in the history of Mathematics. Fermat considered Pythagoras' theorem, which states that, for every right-angled triangle, the square of the hypotenuse is equal to the sum of the Diophantine equations are numerically rich because of their variety. There is no universal method available to know whether a Diophantine equation has a solution or for finding all solutions, if it exists. There are but very few Diophantine problems for each of which the complete solution is known. For example, it is possible to derive all the triplets of integers (x,y,z) that satisfy the equation $x^2+y^2=z^2$. There are several Diophantine equations that have no solutions, trivial solutions, finitely many solutions or an infinite number of solutions. For example, $xy=x+2y+2$ has finite number of solutions and they are $(0,-1),(-2,0),(1,-3),(3,5),(4,3),(6,2)$. The binary quadratic equation $2(x+y)+xy=x^2-y^2$ representing a hyperbola has infinitely many solutions. The Pellian equation $y^2=3x^2-1$ has no solution in integers.

This communication concerns with two negative Pell's equations $x^2=13y^2-17^t$ and $x^2=113y^2-97^t$, where $t\in N$ and infinitely many positive integer solutions are obtained for the choices of t given by

(i) $t=1$

(ii) $t=3$,

(iii) $t=5$,

(iv) $t=2k$,

(v) $t=2k+5$.

A few interesting relations among the solutions are presented. Further recurrence relations on the solutions are derived.

2. Method of Analysis

2.1 Diophantine Equation $x^2=13y^2-17^t$

In this section concerns with the negative Pell's equation $x^2=13y^2-17^t, t\in N$, and infinitely many positive integer solutions are obtained for the choices of v given by

(i) $t=1$

(ii) $t=3$,

(iii) $t=5$,

(iv) $t=2k$,

(v) $t=2k+5$.

A few interesting relations among the solutions are presented. Further recurrence relations on the solutions are derived.

Proposition 2.1. *Let p be a prime. The negative Pell's equation*

$$x^2-py^2=-1$$

is solvable if and only if $p=2$ or $p\equiv 1(\bmod 4)$.

Proof. First we concerns with a negative Pell equation

$$x^2=13y^2-1$$

Here we consider the prime 13 which confirms the existence of integer solutions of using above Proposition 2.1. □

Choice $1: t=1$
The Pell equation is

$$x^2=13y^2-17 \tag{2.1}$$

Let (x_0,y_0) be the initial solution of (2.1) given by $x_0=10;\quad y_0=3$ To find the other solutions of (2.1), consider the Pell equation

$$x^2=13y^2+1$$

whose initial solution $(\widetilde{x}_n,\widetilde{y}_n)$ is given by

$$\widetilde{x}_n=\frac{1}{2}f_n$$
$$\widetilde{y}_n=\frac{1}{2\sqrt{13}}g_n$$

where

$$f_n=(649+180\sqrt{13})^{n+1}+(649-180\sqrt{13})^{n+1}$$
$$g_n=(649+180\sqrt{13})^{n+1}-(649-180\sqrt{13})^{n+1},$$

for, $n=0,1,2,\cdots$. Applying Brahma Gupta lemma between (x_0,y_0) and $(\widetilde{x}_n,\widetilde{y}_n)$, the sequence of non $-$ zero distinct integer solutions to (2.1) are obtained as

$$x_{n+1}=\frac{1}{2}\left[10f_n+3\sqrt{13}g_n\right] \tag{2.2}$$
$$y_{n+1}=\frac{1}{2\sqrt{13}}\left[3\sqrt{13}f_n+10g_n\right] \tag{2.3}$$

The recurrence relation satisfied by the solutions of (2.1) are given by

$$x_{n+2}-1298x_{n+1}+x_n=0$$
$$y_{n+2}-1298y_{n+1}+y_n=0$$

2.2 Diophantine Equation $x^2=113y^2-97^t$

In this section concerns with the negative Pell's equation $x^2=113y^2-97^t$, where $t\in N$, and infinitely many positive integer solutions are obtained for the choices of t given by

(i) $t=1$

(ii) $t=3$,

(iii) $t=5$,

(iv) $t=2k$,

(v) $t=2k+5$.

A few interesting relations among the solutions are presented. Further recurrence relations on the solutions are derived. Here we consider the prime 113 which confirms the existence of integer solutions of using above Proposition 2.1.

Choice $1 : t = 1$

The Pell equation is

$$x^2 = 113y^2 - 97 \tag{2.4}$$

Let (x_0, y_0) be the initial solution of (2.4) given by $x_0 = 4;\quad y_0 = 1$. To find the other solutions of (2.4), consider the Pell equation

$$x^2 = 113y^2 + 1$$

whose initial solution $(\widetilde{x}_n, \widetilde{y}_n)$ is given by

$$\begin{aligned}\widetilde{x}_n &= \frac{1}{2} f_n \\ \widetilde{y}_n &= \frac{1}{2\sqrt{113}} g_n\end{aligned}$$

where

$$\begin{aligned}f_n &= (1204353 + 113296\sqrt{113})^{n+1} + (1204353 - 113296\sqrt{113})^{n+1} \\ g_n &= (1204353 + 113296\sqrt{113})^{n+1} - (1204353 - 113296\sqrt{113})^{n+1}\end{aligned}$$

for, $n = 0, 1, 2, \cdots$. Applying Brahma Gupta lemma between (x_0, y_0) and $(\widetilde{x}_n, \widetilde{y}_n)$, the sequence of non - zero distinct integer solutions to (2.4) are obtained as

$$x_{n+1} = \frac{1}{2}\left[4f_n + \sqrt{113}g_n\right] \tag{2.5}$$

$$y_{n+1} = \frac{1}{2\sqrt{113}}\left[\sqrt{113}f_n + 4g_n\right] \tag{2.6}$$

The recurrence relation satisfied by the solutions of (2.4) are given by

$$\begin{aligned}x_{n+2} - 2408706x_{n+1} + x_n &= 0 \\ y_{n+2} - 2408706y_{n+1} + y_n &= 0\end{aligned}$$

Choice 4: $t = 2k, k > 0$

The Pell equation is

$$x^2 = 113y^2 - 97^{2k}, k > 0 \tag{2.7}$$

Let (x_0, y_0) be the initial solution of (2.7) given by

$$x_0 = 97^k.776; \quad y_0 = 97^k.73$$

Applying Brahma Gupta lemma between (x_0, y_0) and $(\widetilde{x}_n, \widetilde{y}_n)$, the sequence of non $-$ zero distinct integer solutions to (2.7) are obtained as

$$x_{n+1} = \frac{97^\kappa}{2}\left[776f_n + 73\sqrt{113}g_n\right] \tag{2.8}$$

$$y_{n+1} = \frac{97^k}{2\sqrt{113}}\left[73\sqrt{113}f_n + 776g_n\right] \tag{2.9}$$

The recurrence relations satisfied by the solutions of (2.7) are given by

$$\begin{aligned}x_{n+2} - 2408706x_{n+1} + x_n &= 0 \\ y_{n+2} - 2408706y_{n+1} + y_n &= 0\end{aligned}$$

Choice $5 : t = 2k+5, k > 0$

The Pell equation is

$$x^2 = 13y^2 - 97^{2k+5}, k > 0 \tag{2.10}$$

Let (x_0, y_0) be the initial solution of (2.10) given by

$$x_0 = 97^{k-1} \cdot 71450012; \quad y_0 = 97^{k-1} \cdot 6774433$$

Applying Brahma Gupta lemma between (x_0, y_0) and $(\widetilde{x}_n, \widetilde{y}_n)$, the sequence of non - zero distinct integer solutions to (2.10) are obtained as

$$x_{n+1} = \frac{97^{k-1}}{2}\left[71450012f_n + 6774433\sqrt{113}g_n\right] \tag{2.11}$$

$$y_{n+1} = \frac{97^{k-1}}{2\sqrt{113}}\left[6774433\sqrt{113}f_n + 71450012g_n\right] \tag{2.12}$$

The recurrence relations satisfied by the solutions of (2.10) are given by

$$\begin{aligned}x_{n+2} - 2408706x_{n+1} + x_n &= 0 \\ y_{n+2} - 2408706y_{n+1} + y_n &= 0\end{aligned}$$

3. Conclusion

Solving a Pell's equation using the above method provides powerful tool for finding solutions of equations of similar type. Neglecting any time consideration it is possible using current methods to determine the solvability of Pell-like equation.

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