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ON GENERALISED RECURRENT FINSLER SPACES

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Introduction

In one of his papers Vranceanu [9] has defined a non-symmetric connection in an n- dimensional space A_n, we extend this concept to the theory of n- dimensional Finsler space with non-symmetric connection $\Gamma^i_{jk}(\neq \Gamma^i_{kj})$ based on a non-symmetric fundamental tensor $g_{ij}(x,\dot{x})(\neq g_{ji}(x,\dot{x}))$. Let us write

(1.1) $\Gamma^i_{jk} = M^i_{jk} + \frac{1}{2}N^i_{jk}$,

Where M^i_{jk} and $\frac{1}{2}N^i_{jk}$ are respectively the symmetric and skew-symmetric parts of Γ^i_{jk} Following Cartan [1], let a vertical stroke (|) followed by an index denote covariant derivative with respect to x, here we define the covariant derivative of any contravariant vector field $X^i(x,\dot{x})$ as follows:

(1.2) $X^{i}\,{}_{+}|_{j} = \partial j X^i - (\dot{\partial}_m X^i)\ \Gamma^m_{kj}\ \dot{x}^k + X^k\ \Gamma^i_{kj}$

where, a positive sign below an index and followed by a vertical stroke indicates that the covariant derivative has been formed with respect to the connection Γ^i_{jk} as for as that index is concerned. The covariant derivative defined in (1.2) will be called ⊕- covariant differentiation of $X^i(x,\dot{x})$ with respect to $\dot{x}^j$. Differentiating (1.2) ⊕- covariantly with respect to x^k and taking the skew-symmetric part of the result so obtained with respect to indices j and k, we obtain the following commutation formula

(1.3) $X^{i}\,{}_{+}|_{jk} - X^{i}\,{}_{+}|_{kj} = -(\dot{\partial}_m X^i)\ R^m_{pjk}\ \dot{x}^p + X^m\ R^i_{mjk} + X^{i}\,{}_{+}|_{m}\ N^m_{kj}$

where (1.4) ${}^{+}R^h_{ijk} \overset{\text{def}}{=} \partial_k\Gamma^h_{ij} - \partial_j\Gamma^h_{ik} + \dot{\partial}_m\Gamma^h_{ik}\ \Gamma^m_{sj}\ \dot{x}^s - \dot{\partial}_m\Gamma^h_{ij}\ \Gamma^m_{sk}\ \dot{x}^s + \Gamma^p_{ij}\ \Gamma^h_{pk} - \Gamma^p_{ik}\ \Gamma^h_{pj}$.

The entities R^h_{ijk} defined by (1.4) is called " Curvature Tensor" of the Finsler space F_n equipped with non-symmetric connection. From here onwards the Finsler space equipped with non-symmetric connection will be denoted by F^*_n. We shall extensively use the following identities, notations and contractions:

(1.5) (a) $x^{i+}|_{k} = \dot{x}^{i-}|_{k} = 0$, (b) $R^i_{jk} = R^i_{hjk}\ \dot{x}^h$, (c) $R^i_j = R^i_{hj}\ \dot{x}^h$,
(d) $R^i_{hjk} = -R^i_{hkj}$, (e) R^i_i = (n-1) R,
(f) $N^i_{jk} = -N^i_{kj} = \Gamma^i_{jk} - \Gamma^i_{kj}$, (g) $\Gamma^i_{hjk} = \dot{\partial}_h\Gamma^i_{jk}$.

The projective change in F_n is given by

(1.6) $\bar{G}^i(x,\dot{x}) = G^i(x,\dot{x}) - P(x,\dot{x})\dot{x}^i$,

where $P(x,\dot{x})$ are homogeneous scalar functions of degree one in directional arguments $\dot{x}^i$ Therefore, the function $\bar{G}^i(x,\dot{x})$ are also homogeneous of degree two in their directional arguments.
Douglas [2] deduced the entities

(1.7) $\Pi^i_{jk}(x,\dot{x}) \overset{\text{def}}{=} G^i_{jk} - \frac{1}{n+1}\left\{2\delta^i_{(j}\ G^r_{k)r} + \dot{x}^i G^r_{rjk}\right\}$, which are invariant under the projective change (1.6). These entities are called "coefficients of projective connection". They are symmetric in their lower indices and are homogeneous of degree zero in their directional arguments.

Mishra [4] has defined the projective covariant derivative with respect to x^k for these connection parameters in the following form

(1.8) $T^i_{j((k))} = \partial_k T^i_j - (\dot{\partial}_m T^i_j)\, \Pi^m_{pk}\, \dot{x}^p + T^m_j \Pi^i_{mk} - T^i_m \Pi^m_{jk}$.

By the repeated application of the covariant differentiation process as has been given in (1.8) and applying the process of covariant differentiation therefore, We have the following commutation formula

(1.9) $2\, X^i_{[((h))((k))]} = -(\dot{\partial}_r X^i)\, Q^m_{pkh}\, \dot{x}^p + X^m Q^i_{mkh}$,

where (1.10) $Q^i_{jkh}(x, \dot{x}) \overset{\text{def}}{=} 2\{\dot{\partial}_{[h} \Pi^i_{k]j} - (\dot{\partial}_r \Pi^i_{j[s}) \Pi^r_{h]s}\, \dot{x}^s + \Pi^r_{j[k} \Pi^i_{h]r}\}$,

Q^i_{jkh} are called "**projective entities**" and can also be expressed as

(1.11) $Q^i_{jkh} = H^i_{jkh} + \frac{1}{n+1}(\delta^i_j H^r_{rhk} + \dot{x}^i \dot{\partial}_r H^r_{rhk}) + \frac{2}{(n+1)^2}\{(n+1)\, G^r_{rj[(k)} \delta^i_{h]} +$

$+ \delta^i_{[h} \dot{\partial}_{k]} (G^r_{rj} G^s_s)\}$.

The projective entities defined as above, do not form the component of a tensor, however, there are homogeneous of degree one in their directional arguments.

Generalised 2- $\underline{R^+}$- Recurrent $\underline{F^*_n}$

Definition(2.1):

An F^*_n is said to be R^+- recurrent of first order if the curvature tensor ${}^+R^h_{ijk}(x, \dot{x})$ satisfies

(2.1) ${}^+R^h_{ijk}{}^+|_{\,l} = \lambda_l\, {}^+R^h_{ijk}$,

where $\lambda_l = \lambda_l(x, \dot{x})$ is the non-null recurrence vector.

Deffinition (2.2):

A Finsler space F^*_n is said to be generalized 2 - R^+ - recurrent if the curvature tensor ${}^+R^h_{ijk}$ satisfies the relation

(2.2) ${}^+R^h_{ijk}{}^+|_{\,lm} = \beta_m\, {}^+R^h_{ijk}{}^+|_{\,l} + a_{lm}\, {}^+R^h_{ijk}$,

where β_m and a_{lm} are respectively called associate vector and associate tensor of recurrence, it should also be noted here that a_{lm} is non-symmetric.

We now apply a commutator to the indices l and m in (2.2) and then use the commutation formula given by (1.3) and get

(2.3) $- {}^+R^h_{rijk}\, {}^+R^r_{lm} + {}^+R^r_{ijk}\, {}^+R^h_{rlm} - {}^+R^h_{rjk}\, {}^+R^r_{ilm} - {}^+R^h_{irk}\, {}^+R^r_{jlm} + N^r_{ml}({}^+R^h_{ijk}{}^+|_{\,r})$ -

$- {}^+R^h_{ijr}\, {}^+R^r_{kml} = \beta_m\, {}^+R^h_{ijk}{}^+|_{\,l} + (a_{ml} - a_{lm})\, {}^+R^h_{ijk}$.

From (2.3) it can easily be observed that a_{lm} is non-symmetric.

If it be assumed that a_{lm} is a symmetric recurrence tensor and ${}^+R^h_{ijk}$ is a first order recurrent curvature tensor with respect to given associate vector of recurrence β_m then the identity (2.3) reduces into the following alternative form

(2.4) $- {}^{+}R^{h}_{rijk}\,{}^{+}R^{r}_{lm} + {}^{+}R^{r}_{ijk}\,{}^{+}R^{h}_{rlm} - {}^{+}R^{h}_{rjk}\,{}^{+}R^{r}_{ilm} - {}^{+}R^{h}_{irk}\,{}^{+}R^{r}_{jlm} - {}^{+}R^{h}_{ijr}\,{}^{+}R^{r}_{klm} +$
$+ \beta_r\,{}^{+}R^{h}_{ijk}\,N^{r}_{ml} = 0$.

Contracting (2.4) with respect to the indices h and k and thereafter using (1.5), we get
(2.5) $- (\dot{\partial}_r\,{}^{+}R_{ij})\,{}^{+}R^{r}_{lm} - {}^{+}R_{rj}\,{}^{+}R^{r}_{ilm} - {}^{+}R_{ir}\,{}^{+}R^{r}_{jlm} + \beta_r\,{}^{+}R_{ij}\,N^{r}_{ml} = 0$.

With the help of (1.5), we can obtain the following identities
(2.6) (a) ${}^{+}R_{k} = \dot{x}^{r}\,{}^{+}R_{jk}$ (b) ${}^{+}R_{ij}\,\dot{x}^{i}\dot{x}^{j} = {}^{+}R_{j}\,\dot{x}^{j} = (n-1)\,{}^{+}R$,
(c) $(\dot{\partial}_r\,{}^{+}R_{j})\,\dot{x}^{j} = (n-1)\,\dot{\partial}_r\,{}^{+}R - {}^{+}R_r$.

Transverting (2.5) by $\dot{x}^{i}$ and $\dot{x}^{j}$ successively and then using (2.6), we get
(2.7) $(\dot{\partial}_r\,{}^{+}R)\,{}^{+}R^{r}_{lm} = \beta_r\,{}^{+}R\,N^{r}_{ml}$.

Therefore, we can state:

Theorem(2.1)
In a generalized 2 - ${}^{+}$R recurrent F^{*}_{n} (2.7) always holds good provided a_{lm} is symmetric and ${}^{+}R^{h}_{ijk}$ be supposed to satisfy first order recurrency condition as has given by (2.1).

We now transvect (2.2) by $\dot{x}^{i}$ and thereafter using (1.5b), we get
(2.8) ${}^{+}R^{h}_{jk}\,{}^{+}|_{lm} = \beta_m\,{}^{+}R^{h}_{jk}\,{}^{+}|_{l} + a_{lm}\,{}^{+}R^{h}_{jk}$.

Commutating (2.8) with respect to the indices l and m and then using (1.3), we get
(2.9) $- (\dot{\partial}_r\,{}^{+}R^{h}_{jk})\,{}^{+}R^{r}_{lm} + {}^{+}R^{r}_{jk}\,{}^{+}R^{h}_{rlm} - {}^{+}R^{h}_{rk}\,{}^{+}R^{r}_{jlm} - {}^{+}R^{h}_{jr}\,{}^{+}R^{r}_{klm} + {}^{+}R^{h}_{jk}\,{}^{+}|_{r}\,N^{r}_{ml}$
$= \beta_l\,{}^{+}R^{h}_{jk}\,{}^{+}|_{m} + (a_{lm} - a_{ml})\,{}^{+}R^{h}_{jk}$

Differentiating (2.9), $\oplus$ - covariantly with respect to x^{p} and then transvecting it by $\dot{x}^{j}$, we get the following identity in view of (1.5)
(2.10) $(a_{lm} - a_{ml})\,{}^{+}|_{p}\,R^{h}_{k} + (a_{lm} - a_{ml})R^{h}_{k}\,{}^{+}|_{p} + R^{h}_{k}\,{}^{+}|_{lp}\,\beta_n - R^{h}_{k}\,{}^{+}|_{mp}\,\beta_l -$
$- R^{h}_{k}\,{}^{+}|_{m}\,\beta_l\,{}^{+}|_{p} = - (\dot{\partial}_r\,{}^{+}R^{h}_{jk})^{+}|_{p}\,\dot{x}^{j}\,{}^{+}R^{r}_{lm} - (\dot{\partial}_r\,{}^{+}R^{h}_{jk})\,\dot{x}^{j}\,{}^{+}R^{r}_{lm}\,{}^{+}|_{p} +$
$+ {}^{+}R^{h}_{rlm}\,{}^{+}R^{r}_{k}\,{}^{+}|_{p} + {}^{+}R^{r}_{rk}\,{}^{+}R^{h}_{rlm}\,{}^{+}|_{p} - {}^{+}R^{h}_{rk}\,{}^{+}|_{p}\,{}^{+}R^{r}_{lm} - {}^{+}R^{h}_{rk}\,{}^{+}R^{r}_{lm}\,{}^{+}|_{p}$
$- {}^{+}R^{h}_{r}\,{}^{+}|_{p}\,{}^{+}R^{r}_{klm} - {}^{+}R^{h}_{r}\,{}^{+}R^{r}_{klm}\,{}^{+}|_{p} + {}^{+}R^{h}_{k}\,{}^{+}|_{rp}\,N^{r}_{ml} + - {}^{+}R^{h}_{k}\,{}^{+}|_{r}\,N^{r}_{ml}\,{}^{+}|_{p}$.

Contracting (2.10) with respect to the indices h and k and then using (2.3), (1.5), we get
(2.11) ${}^{+}R\,{}^{+}|_{r}\,(\beta_p\,N^{r}_{ml} + N^{r}_{ml}\,{}^{+}|_{p}) + a_{rp}\,N^{r}_{ml} = (n-1)\,[\,\{(a_{lm} - a_{ml})\,{}^{+}|_{p} +$
$+ (\beta_m\,a_{lm} - \beta_l a_{mp})\,{}^{+}R + (a_{lm} - a_{ml})\,{}^{+}R\,{}^{+}|_{p} + (\beta_m\,\beta_p + \beta_m\,{}^{+}|_{p})\,{}^{+}R\,{}^{+}|_{p} -$
$- (\beta_l\,\beta_p + \beta_l\,{}^{+}|_{p})\,{}^{+}R\,{}^{+}|_{p} + \{(\dot{\partial}_r\,{}^{+}R)\,{}^{+}R^{r}_{lm}\}^{+}|_{p}\,]$.

Allowing a cyclic interchange of the indices p, l and m in (2.11) and adding all the three equations thus obtained, we get
(2.12) $({}^{+}R\,{}^{+}|_{r}\,\beta_{[p}\,{}^{+}R\,a_{r[p})\,N^{r}_{ml]} + {}^{+}R\,{}^{+}|_{p}\,N^{r}_{[ml}\,{}^{+}|_{p]} = [\,\{(n-1)\,a_{[lm}\,{}^{+}|_{p]} +$

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$$+ \beta_{[m}\alpha_{lp]} \}^{+}R + \alpha_{[lm}{}^{+}R\,^{+}|_{p]} + \beta_{[m}{}^{+}|_{p}\,{}^{+}R\,^{+}|_{l]} - \beta_{[l}{}^{+}|_{p}\,{}^{+}R\,^{+}|_{m]} +$$
$$+ \{(\dot{\partial}_r{}^{+}R)\,{}^{+}R^{r}_{[lm}\}^{+}|_{p]}\,],$$

where (2.13) $\alpha_{lm} = \alpha_{lm} - \alpha_{ml}$

Therefore, we can state:

Theorem(2.2):

In a generalized 2 - $^{+}$R recurrent F_n^* , the identity given by (2.12) is always true.

Differentiating (2.8) partially with respect to $\dot{x}^i$, we get

$$(2.14)\quad \dot{\partial}_i(\,{}^{+}R^{h}_{jk}{}^{+}|_{p}) = \dot{\partial}_i(\,{}^{+}R^{h}_{jk}{}^{+}|_{l})\beta_m + {}^{+}R^{h}_{jk}{}^{+}|_{l}\,(\dot{\partial}_i\,\beta_m) + (\dot{\partial}_i\,a_{lm})\,{}^{+}R^{h}_{jk} + $$
$$+ a_{lm}(\dot{\partial}_i{}^{+}R^{h}_{jk})$$

Using (1.3), (1.5) and (2.2) in (2.14), we get

$$(2.15)\ \dot{x}^q\,{}^{+}R^{h}_{iqjk}{}^{+}|_{lm} + \{{}^{+}R^{r}_{jk}{}^{+}|_{l}\,\Gamma^{r}_{ijm} - {}^{+}R^{h}_{jr}{}^{+}|_{m}\,\Gamma^{r}_{ikl} - {}^{+}R^{h}_{jr}{}^{+}|_{l}\,\Gamma^{r}_{ikm} -$$
$$- ({}^{+}R^{h}_{rjk}{}^{+}|_{m} + \dot{x}^q\,{}^{+}R^{h}_{rqjk}{}^{+}|_{m})\,\Gamma^{r}_{ipl}\,\dot{x}^p - ({}^{+}R^{h}_{rjk}{}^{+}|_{l} + \dot{x}^q\,{}^{+}R^{h}_{rqjk}{}^{+}|_{m})\,\Gamma^{r}_{ipm}\,\dot{x}^p\} +$$
$$+ \{{}^{+}R^{r}_{jk}\,\Gamma^{h}_{irl}{}^{+}|_{m} - {}^{+}R^{h}_{rk}\,\Gamma^{r}_{ijl}{}^{+}|_{m} - {}^{+}R^{h}_{jr}\,\Gamma^{r}_{ikl}{}^{+}|_{m} - ({}^{+}R^{h}_{rjk} + \dot{x}^q\,{}^{+}R^{h}_{rqjk})\,\Gamma^{r}_{ipl}{}^{+}|_{m}\,\dot{x}^p -$$
$$- ({}^{+}R^{h}_{sjk} + \dot{x}^q\,{}^{+}R^{h}_{sqjk})\,\Gamma^{s}_{rtl}\,\dot{x}^t\,\}\,\Gamma^{r}_{ipm}\,\dot{x}^p$$
$$= \{\beta_m\,\dot{x}^q\,{}^{+}R^{h}_{rqjk}{}^{+}|_{l} + {}^{+}R^{r}_{jk}\Gamma^{h}_{irl} - {}^{+}R^{h}_{rk}\Gamma^{r}_{ijl} - {}^{+}R^{h}_{jr}\,\Gamma^{r}_{ikl} - ({}^{+}R^{h}_{rjk} + \dot{x}^q\,{}^{+}R^{h}_{rqjk})\,\Gamma^{r}_{ipl}\,\dot{x}^p\}$$
$$+ {}^{+}R^{h}_{jk}{}^{+}|_{l}\,\dot{\partial}_i\,\beta_m + (\dot{\partial}_i\,a_{lm})\,{}^{+}R^{h}_{jk} + a_{lm}\,\dot{x}^q\,{}^{+}R^{h}_{iqjk}\ .$$

Commutating (2.15) with respect to the indices l and m, we get

$$(2.16)\quad \dot{x}^q\,\{{}^{+}R^{h}_{iqjk}{}^{+}|_{[lm]} - {}^{+}R^{h}_{sqjk}\,\dot{x}^t\,\Gamma^{s}_{rt[l}\,\Gamma^{r}_{<ip>m]}\,\dot{x}^p - {}^{+}R^{h}_{iqjk}{}^{+}|_{[l}\beta_{m]} -$$
$$- {}^{+}R^{h}_{iqjk}\,a_{[lm]}\,\} = R^{r}_{jk}{}^{+}|_{r}\,\Gamma^{r}_{i[lm]} + \{R^{h}_{sk}\Gamma^{s}_{rj[l} + {}^{+}R^{h}_{js}\,\Gamma^{s}_{rk[l} - {}^{+}R^{s}_{jk}\,\Gamma^{h}_{rs[l} +$$
$$- {}^{+}R^{h}_{sjk}\,\dot{x}^t\,{}^{+}\Gamma^{s}_{rt[i}\}\,\Gamma^{r}_{<ip>m]}\,\dot{x}^p + {}^{+}R^{h}_{jk}\,\dot{\partial}_i a_{[lm]} + {}^{+}R^{h}_{jk}{}^{+}|_{[l}\,\dot{\partial}_{<i>}\,\beta_{m]}.$$

where we have written

$$(2.17)\quad \Gamma^{h}_{ikl}{}^{+}|_{l} = \beta_l\,\Gamma^{h}_{ijk}\quad .$$

Contracting (2.16) with respect to the indices h and k and thereafter using (1.5), we get

$$(2.18)\ \dot{x}^q\{{}^{+}R^{h}_{iqjk}{}^{+}|_{[lm]} - {}^{+}R^{h}_{sqjk}\,\dot{x}^t\,\Gamma^{s}_{rt[l}\,\Gamma^{r}_{<ip>m]}\,\dot{x}^p - {}^{+}R^{h}_{iqjh}{}^{+}|_{[l}\beta_{m]} - {}^{+}R^{h}_{iqjh}\,a_{[lm]}\}$$
$$= \tfrac{1}{2}\,{}^{+}R_j{}^{+}|_{r}\,N^{r}_{lm} + \{{}^{+}R_s\,\Gamma^{s}_{rj[l} + {}^{+}R_{st}\,\dot{x}^t\,{}^{+}\Gamma^{s}_{rt[l}\}\,\Gamma^{r}_{<ip>m]}\,\dot{x}^p +$$
$$+ {}^{+}R_j\,\dot{\partial}_i a_{[lm]} + {}^{+}R_j{}^{+}|_{[l}\,\dot{\partial}_{<i>}\,\beta_{m]}\quad .$$

Thereafter, we can state:

Theorem (4.3):

In a generalized 2- $^{+}$R- recurrent F_n^* if the connection coefficient Γ^{h}_{ijk} be supposed to be $^{+}$R- recurrent of order one with respect to the associated vector of recurrence then (2.18) is always satisfied.

3. H - And W - Generalised 2 - Recurrent Finsler Spaces:

First of all we given the following definitions which shall be used in the later discussions.

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Definition (3.1):
An n-dimensional Finsler space F_n is said to be H- recurrent of the first order if the curvature tensor $H^i_{jkh}(x,\dot{x})$ satisfies the following relation Kumar [3]

(3.1) $H^i_{hjk((s))} = \lambda_s \; H^i_{hjk}$

where $\lambda_s(x)$ is a recurrence vector field depending only on positional coordinates.

Diffinition(3.2):
An n-dimensional Finsler space F_n is said to be H- recurrent of the second order if its curvature tensor satisfies the following relation

(3.2) $H^i_{hjk((s))((m))} = d_{sm} \; H^i_{hjk}$,

where $d_{sm}(x,\dot{x})$ is a recurrence tensor field.

Diffinition(3.3):
A Finsler space F_n is said to be H – generalized 2- recurrent if the curvature tensor $H^i_{jkh}(x,\dot{x})$ satisfies the relation

(3.3) $H^i_{jkh((l))((m))} = \mu_m \; H^i_{jkh((l))} + a_{lm} \; H^i_{jkh}$,

where $\mu_m(x)$ and $a_{lm}(x,\dot{x})$ are respectively the associate recurrence vector and associate recurrence tensor.

Diffinition(3.4):
An n-dimensional Finsler space F_n is said to be W – recurrent of first order if the projevtive covariant derivative of $W^i_{hjk}(x,\dot{x})$ satisfies the relation

(3.4) $W^i_{hjk((l))} = \lambda_l \; W^i_{hjk}$,

where $\lambda_l(x)$ is a recurrence vector.

Diffinition(3.5):
A Finsler space F_n is said to be W – generalized 2- recurrent if the projevtive deviation tensor field $W^i_{hjk}(x,\dot{x})$ satisfies the relation

(3.5) $W^i_{hjk((l))((m))} = \gamma_m \; W^i_{hjk((l))} + c_{lm} \; W^i_{hjk}$, $\quad W^i_{hjk} \neq 0$,

where $\gamma_m(x)$ and $c_{lm}(x,\dot{x})$ are respectively the associate vector and associate tensor of recurrence.
Commutating (3.3) with respect to the indices l and m , we get

(3.6) $H^i_{jkh[((l))((m))]} = \mu_m \; H^i_{jkh} \, a_{[lm]} + H^i_{jkh[((l))}\mu_{m]}$.

Applying the commutation formula (1.9) in (3.6), we get

(3.7) $(a_{lm} - a_{ml})\, H^i_{jkh} = 2\, H^i_{jkh[((l))}\mu_{m]} - (\dot{\partial}_r H^i_{jkh})\, Q^r_{lm} + H^r_{jkh}\, Q^i_{rlm} - H^i_{rkh} Q^r_{jlm} -$
$- H^i_{jrh}\, Q^r_{klm} - H^i_{jkr}\, Q^r_{hlm}$.

Equation (3.7) enables us to state that the associate recurrence tensor a_{lm} is non-symmetric. In view of the commutation formula (1.9) taking the projective covariant derivative of (3.7) with respect to x^s and thereafter using the equations (3.1) and (3.7), we get

(3.8) $H^i_{jkh}\,[(a_{lm} - a_{ml})_{((s))} + (\lambda_l\mu_m - \lambda_m\mu_l)_{((s))}] + [(\dot\partial_r H^i_{jkh})\, Q^r_{lm((s))} -$
$- H^r_{jkh}\, Q^i_{rlm((s))} + H^i_{rkh} Q^r_{jlm((s))} + H^i_{jrh}\, Q^r_{klm((s))} + H^i_{jkr}\, Q^r_{hlm((s))}]$
$= - Q^r_{lm}\,[\, H^p_{hjk}\, \Pi^i_{prs} - H^i_{prh}\, \Pi^p_{jrs} - H^i_{jpk}\, \Pi^p_{krs} - H^i_{jkp}\, \Pi^p_{hrs}\,]$.

Transvecting (3.8) by $\dot x^s$ and thereafter using the relation (3.3), we get

(3.9) $\dot x^s[(a_{lm} - a_{ml})_{((s))} + (\lambda_l\mu_m - \lambda_m\mu_l)_{((s))} + (\dot\partial_r H^i_{jkh})\, Q^r_{lm((s))} -$
$- H^r_{jkh}\, Q^i_{rlm((s))} + H^i_{rkh} Q^r_{jlm((s))} + H^i_{jrh}\, Q^r_{klm((s))} + H^i_{jkr}\, Q^r_{hlm((s))}] = 0$.

Therefore, we can state:

Theorem(3.1):

In a H – generalized 2- recurrent Finsler space F_n , if the recurrence vector λ_s be supposed to be independent of directional arguments then (3.9) always holds.

In view of the commutation formula (1.9) taking the projective covariant derivative of (3.8) with respect to x^n and thereafter using (3.1) and (3.8) itself, we get

(3.10) $H^i_{jkh}\,[(a_{lm} - a_{ml})_{((s))((n))} + (\lambda_l\mu_m - \lambda_m\mu_l)_{((s))((n))}]$
$= - [(\dot\partial_r H^i_{jkh})\, Q^r_{lm((s))((n))} - H^r_{jkh}\, Q^i_{rlm((s))((n))} + H^i_{rkh} Q^r_{jlm((s))((n))} -$
$- H^i_{jrh}\, Q^r_{klm((s))((n))} - H^i_{jkr}\, Q^r_{hlm((s))((n))} + 2Q^r_{lm((s))}(\, \Pi^i_{nrp}\, H^p_{jkh} -$
$- \Pi^p_{nrj}\, H^i_{pkh} + \Pi^p_{nrk}\, H^i_{jph} + \Pi^p_{nrh}\, H^i_{jkp}) + Q^r_{lm}[\, H^p_{hjk}\, \Pi^i_{prs((n))} -$
$- H^i_{prh}\, \Pi^p_{jrs((n))} - H^i_{jpk}\, \Pi^p_{krs((n))} - H^i_{jkp}\, \Pi^p_{hrs((n))}]$

where, we have taken into account the fact that the recurrence vector λ_s is independent of directional arguments.

Transvectig (3.10) by $\dot x^s$ and thereafter using the fact that $\dot x^k_{((s))} = 0$, we get

(3.11) $\dot x^s[\, H^i_{jkh}\{(a_{lm} - a_{ml})_{((s))((n))} + (\lambda_l\mu_m - \lambda_m\mu_l)_{((s))((n))}\} +$
$+ \{(\dot\partial_r H^i_{jkh})\, Q^r_{lm((s))((n))} - H^r_{jkh}\, Q^i_{rlm((s))((n))} + H^i_{rkh} Q^r_{jlm((s))((n))} -$
$- H^i_{jrh}\, Q^r_{klm((s))((n))} - H^i_{jkr}\, Q^r_{hlm((s))((n))} + Q^r_{lm((s))}(\, H^p_{jkh}\, \Pi^i_{prn} -$
$- H^i_{phk}\, \Pi^p_{jrn} - H^i_{jph}\, \Pi^p_{krn} - H^i_{jkp}\, \Pi^p_{hrn}\}] = 0$.

Therefore, we can state:

Theorem(3.2):

In a H – generalized 2 – recurrent Finsler space F_n the associate recurrence tensor field a_{lm} satisies (3.11).

If the recurrence vector field $\lambda_s(x)$ be assumed to be equal to the associate recurrence vector $\mu_s(x)$ and $a_{lm} = a_{ml}$ then the equation (3.11) reduce into the following form

(3.12) $(\dot\partial_r H^i_{jkh})\, Q^r_{lm} + H^r_{jkh}\, Q^i_{rlm} - H^i_{rkh} Q^r_{jlm} - H^i_{jrh}\, Q^r_{klm} - H^i_{jkr}\, Q^r_{hlm} = 0$.

Contracting (3.12) with respect to indices i and h, we get

(3.13) $(\dot\partial_r H^i_{jki})\, Q^r_{lm} + 2\, H_{r[k} Q^r_{j]m} = 0$.

Therefore, we can state:

Theorem(3.3):

In a H- generalized 2- recurrent Finsler space F_n if the recurrence vector fields be assumed to be equal and $a_{lm} = a_{ml}$ then the equation (3.13) always holds.

Commutating (3.5) with respect to the indices l and m, we get

$$(3.14)\quad (c_{lm} - c_{ml})W^i_{hjk} + \gamma_m W^i_{hjk(l)} - \gamma_l W^i_{hjk(m)} = 2W^i_{hjk((l))((m))} .$$

In view of the commutation formula (1.9) and the equation (3.4) the equation (3.14) can be written in the following alternative form

$$(3.15)\ (c_{lm} - c_{ml})W^i_{hjk} = (\lambda_m\gamma_l - \lambda_l\gamma_m)\, W^i_{hjk} - (\dot{\partial}_r W^i_{hjk})\, Q^r_{lm} + W^r_{hjk}\, Q^i_{rlm} - - W^i_{rjk} Q^r_{hlm} - W^i_{hrk}\, Q^r_{jlm} - W^i_{hjr}\, Q^r_{klm} .$$

Thus, (3.15) enables us to state that in a W – generalized 2 – recurrent Finsler space the associate recurrence tensor field $c_{lm}(x,\dot{x})$ is non-symmetric.

From have onwards we shall obtain the relationship which exists in between the recurrence vector and the associate recurrence tensor. Taking the projective covariant derivative of (3.15) with respect to x^s and thereafter using (1.9), (3.4) and (3.15), we get

$$(3.16)\quad W^i_{hjk}\,\{(c_{lm} - c_{ml})_{((s))} + (\lambda_m\gamma_l - \lambda_l\gamma_m)_{((s))}\}$$
$$= \{- (\dot{\partial}_r W^i_{hjk})\, Q^r_{lm((s))} - W^r_{hjk}\, Q^i_{rlm((s))} + W^i_{rjk} Q^r_{hlm((s))} +$$
$$+ W^i_{hrk}\, Q^r_{jlm((s))} + W^i_{hjr}\, Q^r_{klm((s))}\} - Q^r_{lm}\,\{ W^p_{hjk}\,\Pi^i_{prs} -$$
$$- W^i_{pjk}\,\Pi^p_{hrs} - W^i_{hpk}\,\Pi^p_{jrs} - W^i_{hjp}\,\Pi^p_{krs}\}$$

where, we have taken into account the fact that the recurrence tensor λ_s is independent of directional arguments.

Transvecting (3.16) by $\dot{x}^s$ and using the fact that $W^i_{hjk} \neq 0$, we get

$$(3.17)\quad \dot{x}^s\,[(c_{lm} - c_{ml})_{((s))} + (\lambda_m\gamma_l - \lambda_l\gamma_m)_{((s))}] + (\dot{\partial}_r W^i_{hjk})\, Q^r_{lm((s))} -$$
$$- W^r_{hjk}\, Q^i_{rlm((s))} + W^i_{rjk} Q^r_{hlm((s))} + W^i_{hrk}\, Q^r_{jlm((s))} + W^i_{hjr}\, Q^r_{klm((s))} = 0\ .$$

Therefore, we can state:

Theorem(3.4)

In a W- generalized 2 – recurrence Finsler space F_n if the recurrence vector λ_l be supposed to be independent of directional arguments then (3.17) is always true.

Differentiating (3.16) projective covariantly with respect to x^n and thereafter using (3.4) and (3.16) itself, we get

$$(3.18)\quad W^i_{hjk}\,\{(c_{lm} - c_{ml})_{((s))((n))} + (\lambda_m\gamma_l - \lambda_l\gamma_m)_{((s))((n))}\}$$
$$= [- (\dot{\partial}_r W^i_{hjk})\, Q^r_{lm((s))((n))} - W^r_{hjk}\, Q^i_{rlm((s))((n))} + W^i_{rjk} Q^r_{hlm((s))((n))} +$$
$$+ W^i_{hrk}\, Q^r_{jlm((s))((n))} + W^i_{hjr}\, Q^r_{klm((s))((n))}\} + 2Q^r_{lm[((s))}\{\,\Pi^i_{n]rp}\, W^p_{hjk} -$$
$$- \Pi^p_{n]rh} W^i_{pjk} - \Pi^p_{n]rj} W^i_{hpk} - \Pi^p_{n]rk} W^i_{hjp}\} + Q^r_{lm[((s))}\{\, W^p_{hjk}\,\Pi^i_{prs((n))} -$$

$$- W^i_{pjk}\ \Pi^p_{hrs((n))} - W^i_{hpk}\ \Pi^p_{jrs((n))} - W^i_{hjp}\ \Pi^p_{krs((n))}\}].$$

Transvecting (3.18) by $\dot{x}^s$ and thereafter using the fact that $\dot{x}^i_{((k))} = 0$, we get

$$(3.19)\quad \dot{x}^s\ [\ W^i_{hjk}\ \{(c_{lm} - c_{ml})_{((s))((n))} + (\lambda_m\gamma_l - \lambda_l\gamma_m)_{((s))((n))}\} +$$

$$+ \{\ (\dot{\partial}_r W^i_{hjk})\ Q^r_{lm((s))((n))} - W^r_{hjk}\ Q^i_{rlm((s))((n))} + W^i_{rjk} Q^r_{hlm((s))((n))} +$$

$$+ W^i_{hrk}\ Q^r_{jlm((s))((n))} + W^i_{hjr}\ Q^r_{klm((s))((n))} + Q^r_{lm((s))}(\ W^p_{hjk}\ \Pi^i_{prm} -$$

$$- W^i_{pjk}\ \Pi^p_{hrm} - W^i_{hpk}\ \Pi^p_{jrm} - W^i_{hjp}\ \Pi^p_{krm})\}] = 0\ .$$

Therefore, we can state:

Theorem(3.5)

In a W- generalized 2 - recurrent Finsler space the associate recurrence tensor field always satisfied (3.19).

If we now assume that the two associate recurrence vectors $\lambda_m(x)$ and $\gamma_m(x)$ are equal and also that the recurrence tensor field $c_{lm}(x, \dot{x})$ is symmetric then from (3.15), we get

$$(3.20)\quad (\dot{\partial}_r W^i_{hjk})\ Q^r_{lm} - W^r_{hjk}\ Q^i_{rlm} + W^i_{rjk} Q^r_{hlm} + W^i_{hrk}\ Q^r_{jlm} + W^i_{hjr}\ Q^r_{klm} = 0\ .$$

Therefore, we can state:

Theorem(3.6)

In a generalized 2 – recurrent Finsler space if the associated recurrence vector $\gamma_l(x)$ be assumed to be equal to the recurrence vector $\lambda_l(x)$ and also the recurrence tensor $c_{lm}(x, \dot{x})$ be assumed to be symmetric then (3.20) always holds.

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