Journal of the Calcutta Mathematical Society, **20,** (1) 55–68 (2024)

COMMON FIXED POINTS RESULTS SATISFYING (ψ, φ)-WEAK CONTRACTION IN PARTIALLY ORDERED b-METRIC SPACES

MEGHA PINGALE[1], RENU PATHAK[2] AND NAVEEN MANI[3]

(*Received 9 April 2023*)

Abstract : The primary objective of the current work is to use (ψ, φ)- weak contraction to prove the existence of coincidence points and establish the uniqueness of common fixed points for two pairs of self-maps in partially ordered $b-$ metric spaces. To ensure the existence of coincidence point, out of two, one of the pairs preserve the b-EA property, and to obtain the uniqueness of the fixed point, they hold a weakly compatible property. An example and few corollaries are given to illustrate the main finding.

Keywords : Fixed points,(ψ, φ)-weak contraction,b-metric sapces

2020 AMS Classification : 47H10, 46TXX, 54H25

I. Introduction. M. Frechet (1906) developed the well-known idea of metric space as an extension to conventional distance. In the theory of metric space, particularly in non-linear analysis, number of authors studied non-contraction mappings. It is proficient that physical problems generally involve nonlinear differential and integral equations. Banach contraction principle (Banach, 1922) plays a vital role to deal with such kind of physical problems and provide a powerful tool for obtaining the solutions of these equations. In general, contraction mappings are continuous. It is a most essential result in the metric fixed point theory, and has several applications and extensions. In 1968, Kannan (1968) proved an extension of Banach (1922) without assuming the continuity condition of the map. Since then, there exist several extensions and generalization of contraction principle, some of them are refer to (Rhoades, 2001, Gupta, Mani and Tripathi, 2012 and Gupta and Mani, 2013). Jungck (1976, 1986, 196) led the idea of commuting, compatible mapping and weak compatible mappings to deduce the fixed point results for pair of self mappings in usual metric spaces. Some work on these types of mappings can be found in (Sessa, 1982, Pant, 1996, Morales, Rojas and Bisht, 2014 and Morales and Rojas, 2016).

The idea of metric spaces is further explored in many different ways in the literature, in addition to the contraction mappings. One of the well known generalizations of metric spaces are b-metric spaces. The idea of b-metric was initiated from the works of Bourbaki (1974) and Bakhtin (1989). Later, Czerwik (1993) introduced and formally defined the notion of b-metric space. In 1998, Czerwik (1998) proved the contraction mapping theorem in sense of $b-$ metric spaces. One thing to keep in mind is that the class of b-metric spaces is essentially larger than that of metric spaces. For more examples, fixed point results, coincidence point results and their applications, definitions of notions as b-convergence, b-completeness, b-Cauchy, and related result in the setting of partially ordered b-metric spaces, we refer (Ozturk and Turkoglu, 2015).

The notion of weak contraction was formally introduced by Alber and Delabriere (1997) in 1997, and is known as ϕ-weak contraction. They proved the existence of fixed points for single-valued maps satisfying weak contraction condition on a closed convex sets of Hilbert spaces. Rhoades (2001) showed that their result is also valid in complete metric spaces. Some further generalization of weak contractions and generalized weak contractions using ψ and ϕ mappings in metric spaces can be found in (Aamri and Moutawakil, 2022 and Aghajani, Abbas and Roshan, 2014).

Several authors studied the existence of fixed point for weak contraction and generalized contractions in the sense of partially ordered sets. The first result in this direction was given by Ran and Reurings (2004) in 2004. In continuation, Nieto and Lopez (2005, 2007) further refined and extended above results with the help of non-decreasing functions and then proved some fixed-point results in such spaces. Recently, Gupta et al. (2016, 2017) proved several fixed point theorems under partially ordered settings by defining some generalized contractions. Some more results on partially ordered b-metric spaces can be found in (Mitiku, Karusala and Namana,, 2020, Delbosco, 1976–1977, Skof, 1977, Khan, Swalech and Sessa, 1984, Dulta and Choudhurry, 2008, Mani, 2018 and Gupta, Jungck and Mani, 2020). Aamri and Moutawakil (2022) introduced the notion of $(E.A)$-property in metric space. Later in 2015, Ozturk and Turkoglu (2015) extended this idea in the setting of b-metric space and give the notion of b-$(E.A)$ property.

Before proceeding to the main results of this paper, lets recall some basic definition, examples and fundamental lemmas that will be quite useful in proving our main theorem.

2. Primitive Concepts and Relevant Literature. Authors in (Bakhtin, 1989 and Czerwik, 1993) defined b-metric space as follows:

DEFINITION 2.1 ***(b-metric space)*** *Let Δ be a space. let R^+ denotes the set of all nonnegative numbers. A function $d : \Delta \times \Delta \to R^+$is said to be an b-metric on Δ if for all x, y, z in Δ and $s \geq 1$, following conditions are satisfied:*

1. $d(x, y) = 0$ *if and only if* $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$

The pair (Δ, d) is called a b-metric space.

EXAMPLE 2.2 *Let (Δ, d) be a metric space and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $s = 2^{p-1}$.*

EXAMPLE 2.3 *Let $\Delta = \{0, 1, 2\}$ define $d : \Delta \times \Delta \to R^+$it as follows*

$$\begin{aligned} d(0,0) &= d(1,1) = d(2,2) = 0; \\ d(1,2) &= d(2,1) = d(0,1) = d(1,0) = 1, \\ d(2,0) &= d(0,2) = m \geq 2 for\ k = \frac{m}{2} \quad where \quad m \geq 2 \end{aligned}$$

Thus, (Δ, d) is a b-metric space but not a metric for $m > 2$.

LEMMA 2.4 *Let (Δ, d) be b-metric Space, and $s \geq 1$ when a sequence $\{x_n\}$ satisfies the following condition:*

$$d(x_n, x_{n+1}) \leq K d(x_{n-1}, x_n)$$

for some $0 < K < \frac{1}{s}$ and $n = 1, 2, 3 \ldots$ Then $\{x_n\}$ is b-Cauchy sequence in (Δ, d).

DEFINITION 2.5 *Let (Δ, d) is a b-metric space and ϖ, ς be the self - mappings defined on Δ. Then*

1. ϖ, ς *are said to be compatible* (Jungck, 1986) *if whenever a sequence $\{x_n\}$ in Δ is such that $\{\varpi x_n\}$ and $\{\varsigma x_n\}$ are b-convergent to some $t \in \Delta$ then*

$$\lim_{n \to \infty} (\varpi \varsigma x_n, \varsigma \varpi x_n) = 0$$

2. ϖ, ς *are said non-compatible* (Jungck, 1986)*, if at least one sequence in* Δ *is such that* $\{\varpi x_n\}$ *and* $\{\varsigma x_n\}$ *are b-convergent to some* $t \in \Delta$ *but*

$$\lim_{n\to\infty} (\varpi\varsigma x_n, \varsigma\varpi x_n)$$

is either nonzero or does not exist.

3. ϖ, ς *are said to satisfy the b-*$(E.A)$ *property* (Banach, 1922) *if there exists a sequence* $\{x_n\}$ *in* Δ *is such that* $\lim_{n\to\infty} \varpi x_n = \lim_{n\to\infty} \varsigma x_n = t$ *for some* $t \in \Delta$*.*

DEFINITION 2.6 (Alber and Guerre-Delabriere, 1997) *A self-mapping* ϖ *on a metric space* Δ *is called a weak* φ*-contraction if* $\varphi : [0, \infty) \to [0, \infty)$ *is a strictly increasing map with* $\varphi(0) = 0$ *and*

$$d(\varpi x, \varpi y) \leq d(x, y) - \varphi(d(x, y))$$

DEFINITION 2.7 (Nieto and Lopez, 2007) *Suppose* Δ *is a non-empty set and* $\preceq$ *is a partially ordered relation on set* Δ*. Then a map* $\varpi : \Delta \to \Delta$ *is said to be non-decreasing if each* $\varsigma, \vartheta \in \Delta, \varsigma \leq \vartheta$ *implies* $\varpi(\varsigma) \leq \varpi(\vartheta)$*.*

DEFINITION 2.8 (Khan, Swalech and Sessa, 1984) *Let us denote* ψ *as the set of all altering distance function* $\psi : [0, \infty) \to [0, \infty)$ *which satisfies the following property*:

1. ψ *is continuous and not decreasing;*

2. $\psi(t) = 0$ *if and only if* $t = 0$*.*

DEFINITION 2.9 (Morales and Rojas, 2021) *Denote* Φ *as the set of all functions* $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *satisfying the following conditions*:

1. $\varphi(0) = 0$ *and for all* $t > 0$*,* $\varphi(t) > 0$*,*

2. φ *is lower semicontinuous function.*

REMARK 2.10 *Its worth to mention here* $\Psi \subset \Phi$*.*

DEFINITION 2.12 (Morales AND Rojas, 2021) *Let* (Δ, d) *is a b-metric space with* $s \geq 1$ *and let* ϖ, ς *be self-mappings of* Δ*. The mappings* ϖ, ς *are said to be of* (ψ, φ) *-weak contraction type if there exist,* $\psi \in \Psi$ *and* $\varphi \in \Phi$ *such that for all* $, x, y \in \Delta$*,*

$$\psi(sd(\varpi x, \varpi y) \leq \psi(d(\varsigma x, \varsigma y)) - \varphi(d(\varsigma x, \varsigma y)).$$

Authors in (Morales and Rojas, 2021) have given some important remarks and few examples to justify the novality of their result.

The main aim of this work is to prove a fixed-point result using the altering distance functions for the four self-maps that satisfies b-$(E.A.)$ property. Our presented work explicitly generalized some recent results from the existing literature.

3. Main results

THEOREM 3.1 *Let* $(\Delta, \preceq, d)$ *is a partially ordered* $b-$ *metric space with* $s \geq 1$. *Let* $\varpi, \varsigma, \vartheta, \sigma : \Delta \to \Delta$ *be four self mapping with* $\varpi(\Delta) \subseteq \sigma(\Delta), \varsigma(\Delta) \subseteq \vartheta(\Delta)$ *such that for all* $x, y \in \Delta$, *comparable elements* σx, ϑy *satisfies*:

$$\psi\left(s^{\varepsilon} d(\varpi x, \varsigma y)\right) \leq \psi(M(x,y)) - \varphi(M(x,y)), \tag{1}$$

where $\varepsilon > 1$ *is a constant,* ψ, φ *are altering distance functions and*

$$M((x,y)) = \max\left\{\begin{array}{c} d(\vartheta x, \sigma y), d(\varpi x, \vartheta x), d(\varsigma y, \sigma y), \\ \frac{1}{2s}[d(\varpi x, \sigma y) + d(\vartheta x, \varsigma y)], \frac{1}{2}[d(\vartheta x, \sigma y) + d(\varpi x, \vartheta x)] \end{array}\right\}. \tag{2}$$

Suppose that one of the pairs $(\varpi, \vartheta), (\varsigma, \sigma)$ *satisfy the b-(E.A.) property and that one of the subspaces* $\varpi(\Delta), \varsigma(\Delta), \vartheta(\Delta), \sigma(\Delta)$ *is b-closed in* Δ.

Further, if for every non-increasing sequence $\{x_n\}$ *and a sequence* $\{y_n\}$ *with* $y_n \preceq x_n$ *for all* $y_n \to u$, *we have* $u \preceq x_n$. *Then the pairs* $(\varpi, \vartheta), (\varsigma, \sigma)$ *have a coincidence point in* Δ.

Moreover, if the pairs, $(\varpi, \vartheta), (\varsigma, \sigma)$ *are compatible, then* $\varpi, \varsigma, \vartheta, \sigma$ *have a unique common fixed point.*

Proof: If the pair (ϖ, ϑ) satisfies the b-(E.A.) property, then there exists a sequence $\{x_n\}$ in Δ, for some $q \in \Delta$ satisfying,

$$\lim_{n\to\infty} \varpi x_n = \lim_{n\to\infty} \vartheta x_n = q. \tag{3}$$

As $\omega(\Delta) \subseteq \sigma(\Delta)$. there exists a sequence $\{y_n\}$ in Δ, where $y_n \preceq x_n$ for all n, such that

$$\varpi x_n = \sigma y_n$$

Hence

$$\lim_{n\to\infty} \sigma y_n = q \tag{4}$$

To prove $\lim_{n\to\infty} \varsigma y_n = q$.

Since $y_n \preceq x_n$ for all n and also $\sigma y_n = \varpi x_n$, then eq. (1) gives that

$$\psi\left(s^{\varepsilon} d\left(\varpi x_n, \varsigma y_n\right)\right) \leq \psi\left(M\left(x_n, y_n\right)\right) - \varphi\left(M\left(x_n, y_n\right)\right), \tag{5}$$

On using property of ψ and applying limit superior, we get

$$\begin{aligned} \overline{\lim_{n\to\infty}}\, \psi\left(s^{\varepsilon} d\left(\varpi x_n, \varsigma y_n\right)\right) &\leq \overline{\lim_{n\to\infty}}\, s^{\varepsilon} d\left(\varpi x_n, \varsigma y_n\right) \\ &\leq \overline{\lim_{n\to\infty}} \left\{\psi\left(M\left(x_n, y_n\right)\right) - \varphi\left(M\left(x_n, y_n\right)\right)\right\} \end{aligned} \tag{6}$$

where

$$\begin{aligned} M\left(\left(x_n, y_n\right)\right) &= \max\left\{\begin{array}{c} d\left(\vartheta x_n, \sigma y_n\right), d\left(\varpi x_n, \vartheta x_n\right), d\left(\varsigma y_n, \sigma y_n\right), \\ \frac{1}{2s}\left[d\left(\varpi x_n, \sigma y_n\right) + d\left(\vartheta x_n, \varsigma y_n\right)\right], \\ \frac{1}{2}\left[d\left(\vartheta x_n, \sigma y_n\right) + d\left(\varpi x_n, \vartheta x_n\right)\right] \end{array}\right\} \\ &= \max\left\{\begin{array}{c} d\left(\vartheta x_n, \varpi x_n\right), d\left(\varpi x_n, \vartheta x_n\right), d\left(\varsigma y_n, \varpi x_n\right) \\ \frac{1}{2s}\left[d\left(\varpi x_n, \varpi x_n\right) + d\left(\vartheta x_n, \varsigma y_n\right)\right], \\ \frac{1}{2}\left[d\left(\vartheta x_n, \varpi x_n\right) + d\left(\varpi x_n, \vartheta x_n\right)\right] \end{array}\right\} \\ &\leq \max\left\{\begin{array}{c} d\left(\vartheta x_n, \varpi x_n\right), d\left(\varsigma y_n, \varpi x_n\right), d\left(\vartheta x_n, \varpi x_n\right), \\ \frac{1}{2s}\left\{s\left[d\left(\vartheta x_n, \varpi x_n\right) + d\left(\varpi x_n, \varsigma y_n\right)\right]\right\} \end{array}\right\} \\ &\leq \max\left\{\begin{array}{c} d\left(\vartheta x_n, \varpi x_n\right), d\left(\varpi x_n, \varsigma y_n\right), \\ \frac{1}{2}\left[d\left(\vartheta x_n, \varpi x_n\right) + d\left(\varpi x_n, \varsigma y_n\right)\right] \end{array}\right\} \end{aligned}$$

Here, we discuss three possible cases of $M\left(x_n, y_n\right)$ as mentioned below

CASE 1. If $M\left(x_n, y_n\right) = d\left(\vartheta x_n, \varpi x_n\right)$, then eq. (6) implies that

$$\begin{aligned} \overline{\lim_{n\to\infty}}\, \psi\left(s^{\varepsilon} d\left(\varpi x_n, \varsigma y_n\right)\right) &\leq \overline{\lim_{n\to\infty}}\, s^{\varepsilon} d\left(\varpi x_n, \varsigma y_n\right) \\ &\leq \overline{\lim_{n\to\infty}} \left\{\psi\left(d\left(\vartheta x_n, \varpi x_n\right)\right) - \varphi\left(d\left(\vartheta x_n, \varpi x_n\right)\right)\right\} \end{aligned}$$

On using eq. (3) and continuity of ψ, φ give that

$$\overline{\lim_{n\to\infty}}\, s^{\varepsilon} d\left(\varpi x_n, \varsigma y_n\right) \leq 0.$$

This is possible only if $d\left(\varpi x_n, \varsigma y_n\right) = 0$.

Thus $\varsigma y_n \to q$ as $n \to \infty$.

CASE 2. If $M(x_n, y_n) = d(\varpi x_n, \varsigma y_n)$, then again eq. (6) give that

$$\begin{aligned}\overline{\lim_{n\to\infty}}\, \psi\left(s^{\varepsilon} d\left(\varpi x_n, \varsigma y_n\right)\right) &\leq \overline{\lim_{n\to\infty}}\, s^{\varepsilon} d\left(\varpi x_n, \varsigma y_n\right) \\ &\leq \overline{\lim_{n\to\infty}} \left\{\psi\left(d\left(\varpi x_n, \varsigma y_n\right)\right) - \varphi\left(d\left(\varpi x_n, \varsigma y_n\right)\right)\right\}\end{aligned}$$

On using the fact that ψ is continuous and non-decreasing and also φ is continuous, we have

$$\begin{aligned}\overline{\lim_{n\to\infty}}\, s^{\varepsilon} d\left(\varpi x_n, \varsigma y_n\right) &\leq \psi\left(\overline{\lim_{n\to\infty}}\, d\left(\varpi x_n, \varsigma y_n\right)\right) - \varphi\left(\overline{\lim_{n\to\infty}}\, d\left(\varpi x_n, \varsigma y_n\right)\right) \\ &\leq \psi\left(\overline{\lim_{n\to\infty}}\, d\left(\varpi x_n, \varsigma y_n\right)\right) < \overline{\lim_{n\to\infty}}\, d\left(\varpi x_n, \varsigma y_n\right).\end{aligned}$$

As $s^{\varepsilon} > s \geq 1$, it is possible only if $d(\varpi x_n, \varsigma y_n) = 0$.

Thus $\varsigma y_n \to q$ as $n \to \infty$.

CASE 3. If $M(x_n, y_n) = \frac{1}{2}\left[d(\vartheta x_n, \varpi x_n) + d(\varpi x_n, \varsigma y_n)\right]$. Hence from eq. (6), as ψ and φ are continuous functions,

$$\begin{aligned}\overline{\lim_{n\to\infty}}\, \psi\left(s^{\varepsilon} d\left(\varpi x_n, \varsigma y_n\right)\right) &\leq \overline{\lim_{n\to\infty}}\, s^{\varepsilon} d\left(\varpi x_n, \varsigma y_n\right) \\ &\leq \overline{\lim_{n\to\infty}} \left\{\begin{array}{c} \psi\left(\frac{1}{2}\left[d\left(\vartheta x_n, \varpi x_n\right) + d\left(\varpi x_n, \varsigma y_n\right)\right]\right) \\ -\varphi\left(\frac{1}{2}\left[d\left(\vartheta x_n, \varpi x_n\right) + d\left(\varpi x_n, \varsigma y_n\right)\right]\right)\end{array}\right\} \\ &\leq \psi\left(\overline{\lim_{n\to\infty}}\, \frac{1}{2}\left[d\left(\vartheta x_n, \varpi x_n\right) + d\left(\varpi x_n, \varsigma y_n\right)\right]\right) \\ &\qquad -\varphi\left(\overline{\lim_{n\to\infty}}\, \frac{1}{2}\left[d\left(\vartheta x_n, \varpi x_n\right) + d\left(\varpi x_n, \varsigma y_n\right)\right]\right)\end{aligned}$$

Making use of eq. (3) and that fact that ψ is non-decreasing, we get

$$\begin{aligned}\overline{\lim_{n\to\infty}}\, s^{\varepsilon} d\left(\varpi x_n, \varsigma y_n\right) &\leq \psi\left(\overline{\lim_{n\to\infty}}\, \frac{1}{2} d\left(\varpi x_n, \varsigma y_n\right)\right) \\ &\qquad -\varphi\left(\overline{\lim_{n\to\infty}}\, \frac{1}{2} d\left(\varpi x_n, \varsigma y_n\right)\right) \\ &\leq \psi\left(\overline{\lim_{n\to\infty}}\, \frac{1}{2} d\left(\varpi x_n, \varsigma y_n\right)\right) \\ &\leq \overline{\lim_{n\to\infty}}\, \frac{1}{2} d\left(\varpi x_n, \varsigma y_n\right)\end{aligned}$$

Consequently, we get $d(\varpi x_n, \varsigma y_n) = 0$

Since $\sigma(\Delta)$ is a closed subspace Δ, then there exists $r \in \Delta$ such that, $\sigma r = q$.

Next we assert that $\varsigma r = q$. Suppose not, that is $d(q, \varsigma r) \neq 0$ implies that $d(q, \varsigma r) > 0$. therefore by using triangle inequality, we can write

$$0 < \frac{1}{s} d(q, \varsigma r) \leq d\left(q, \varpi x_n\right) + d\left(\varpi x_n, \varsigma r\right) \tag{7}$$

From eq. (1), on substituting $x = x_n$ and $y = r$, we obtain

$$\psi\left(s^{\varepsilon} d\left(\varpi x_n, \varsigma r\right)\right) \leq \psi\left(M\left(x_n, r\right)\right) - \varphi\left(M\left(x_n, r\right)\right), \tag{8}$$

where

$$\begin{aligned} M\left(x_n, r\right) &= \max\left\{\begin{array}{c} d\left(\vartheta x_n, \sigma r\right), d\left(\varpi x_n, \vartheta x_n\right), d(\varsigma r, \sigma r), \\ \frac{1}{2s}\left[d\left(\varpi x_n, \sigma r\right) + d\left(\vartheta x_n, \varsigma r\right)\right], \\ \frac{1}{2}\left[d\left(\vartheta x_n, \sigma r\right) + d\left(\varpi x_n, \vartheta x_n\right)\right] \end{array}\right\} \\ &= \max\left\{\begin{array}{c} d\left(\vartheta x_n, q\right), d\left(\varpi x_n, \vartheta x_n\right), d(\varsigma r, q), \\ \frac{1}{2s}\left[d\left(\varpi x_n, q\right) + d\left(\vartheta x_n, \varsigma r\right)\right], \\ \frac{1}{2}\left[d\left(\vartheta x_n, q\right) + d\left(\varpi x_n, \vartheta x_n\right)\right] \end{array}\right\} \\ &\leq \max\left\{\begin{array}{c} d\left(\vartheta x_n, q\right), d\left(\varpi x_n, \vartheta x_n\right), d(\varsigma r, q), \\ \frac{1}{2s}\left[d\left(\varpi x_n, q\right) + s d\left(\vartheta x_n, q\right) + s d(q, \varsigma r)\right], \\ \frac{1}{2}\left[d\left(\vartheta x_n, q\right) + d\left(\varpi x_n, \vartheta x_n\right)\right] \end{array}\right\} \end{aligned} \tag{9}$$

On letting $\lim n \to \infty$ in eq. (8) and eq. (9), and making use of eq. (3) and the fact ψ is continuous function, we get

$$s^{\varepsilon} d\left(\varpi x_n, \varsigma r\right) \leq d(q, \varsigma r)$$

implies that

$$d\left(\varpi x_n, \varsigma r\right) \leq \frac{1}{s^{\varepsilon}} d(q, \varsigma r)$$

Thus, from (7) on taking $\lim n \to \infty$, we get

$$\begin{aligned} 0 &< \frac{1}{s} d(q, \varsigma r) \leq d\left(q, \varpi x_n\right) + \frac{1}{s^{\varepsilon}} d(q, \varsigma r) \\ 0 &< \frac{1}{s} d(q, \varsigma r) - \frac{1}{s^{\varepsilon}} d(q, \varsigma r) \leq 0 \\ 0 &< \frac{1}{s^{\varepsilon-1}} d(q, \varsigma r) \leq 0 \end{aligned}$$

This is a contradiction to our assumption, and hence $d(q, \varsigma r) = 0$. that is $q = \varsigma r$. Thus we obtain that r the coincidence points of the pair (ς, σ).

As $\varsigma(\Delta) \subseteq \vartheta(\Delta)$, therefore there exists a point $z \in \Delta$ such that $q = \vartheta z$ Next we claim that $\vartheta z = \varpi z$.

Since the elements σx and ϑy are comparable for all $x, y \in \Delta$. Thus via equation (1), we have

$$\psi\left(s^{\varepsilon} d(\varpi z, \varsigma r)\right) \leq \psi(M(z, r)) - \varphi(M(z, r)) \tag{10}$$

where

$$M(z,r) = \max\left\{ \begin{array}{c} d(\vartheta z, \sigma r), d(\varpi z, \vartheta z), d(\varsigma r, \sigma r), \\ \frac{1}{2s}[d(\varpi z, \sigma r) + d(\vartheta z, \varsigma r)], \frac{1}{2}[d(\vartheta z, \sigma r) + d(\varpi z, \vartheta z)] \end{array} \right\}$$

But we know that $\sigma r = \varsigma r = \vartheta z = q$ Above equality implies that

$$\begin{aligned} M(z,r) &= \max\left\{0, d(\varpi z, q), 0, \frac{1}{2s}[d(\varpi z, q) + 0], \frac{1}{2}[0 + d(\varpi z, q)]\right\} \\ M(z,r) &= d(\varpi z, q) \end{aligned}$$

Thus, from equation (10) and using the property of ψ and φ, we have

$$s^{\varepsilon} d(\varpi z, \varsigma r) \leq d(\varpi z, q)$$

It is possible only if $d(\varpi z, \varsigma r) = 0$. Therefore, $\varpi z = \vartheta z = q$. Hence z is the coincidence point of the pair (ϖ, ϑ). Thus, we get that

$$\varpi z = \vartheta z = q = \varsigma r = \sigma r$$

By weak compatibility of pairs (ϖ, ϑ) and (ς, σ) we obtain

$$\varpi q = \vartheta q \quad \text{and} \quad \varsigma q = \sigma q$$

Next we prove that q is a common fixed point of $\varpi, \vartheta, \varsigma, \sigma$

From equation (1), we have,

$$\psi\left(s^{\varepsilon} d(\varpi q, q)\right) = \psi\left(s^{\varepsilon} d(\varpi q, \varsigma r)\right) \leq \psi(M(q, r)) - \varphi(M(q, r)) \tag{11}$$

where

$$M(q,r) = \max\left\{ \begin{array}{c} d(\vartheta q, \sigma r), d(\varpi q, \vartheta q), d(\varsigma r, \sigma r), \\ \frac{1}{2s}[d(\varpi q, \sigma r) + d(\vartheta q, \varsigma r)], \frac{1}{2}[d(\vartheta q, \sigma r) + d(\varpi q, \vartheta r)] \end{array} \right\}$$

On using $\varpi q = \vartheta q, \varsigma q = \sigma q$ and $\varsigma r = \sigma r = q$ in above equality, we get that $M(q,r) = d(\varpi q, q)$. Thus eq. (11), gives that

$$s^{\varepsilon} d(\varpi q, q) \leq \psi(d(\varpi q, q)) - \varphi(d(\varpi q, q)) < \psi(d(\varpi q, q)) < d(\varpi q, q)$$

What it follows that $\varpi q = q = \vartheta q = q$

Similarly one can prove that $\varsigma q = q = \sigma q$. Thus q is a common fixed point of $\varpi, \vartheta, \varsigma, \sigma$.

Further, assume that p is another common fixed point of the maps $\varpi, \varsigma, \vartheta, \sigma$. From Eq. (1), we have

$$\psi\left(s^{\varepsilon} d(q,p)\right) = \psi\left(s^{\varepsilon} d(\varpi q, \varsigma p)\right) \leq \psi(M(q,p)) - \varphi(M(q,p)) \tag{12}$$

where

$$\begin{aligned} M(q,p) &= \max\left\{ \begin{array}{c} d(\vartheta q, \sigma p), d(\varpi q, \vartheta q), d(\sigma p, \varsigma q), \\ \frac{1}{2s}[d(\varpi q, \sigma p) + d(\vartheta q, \varsigma p)], \frac{1}{2}[d(\vartheta q, \sigma p) + d(\varpi q, \vartheta p)] \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} d(q,p), d(q,q), d(p,q), \frac{1}{2s}[d(q,p) + d(q,p)], \\ \frac{1}{2}[d(q,p) + d(q,p)] \end{array} \right\} \\ &= d(q,p) \end{aligned}$$

Eq. (12) implies that

$$\begin{aligned} \psi\left(s^{\varepsilon} d(q,p)\right) &\leq \psi(d(q,p)) - \varphi(d(q,p)) \\ \psi\left(s^{\varepsilon} d(q,p)\right) &\leq \psi(d(q,p)) \\ s^{\varepsilon} d(q,p) &\leq d(q,p) \end{aligned}$$

From which it follows that

$$d(q,p) = 0$$

Hence $q = p$. This proved our result.

EXAMPLE 3.2 *Let* $\Delta = [0,1]$ *with* $s = 2 > 1$, *and let* $d(x,y) = |x-y|^2$ *be a metric. Then clearly the triplet* $(\Delta, \preceq, d)$ *is a partially ordered* $b-$ *metric space with* $x > y$.

Define the maps $\varpi, \varsigma, \vartheta, \sigma : \Delta \to \Delta$ *as*

$$\varpi x = \frac{x}{12}, \sigma x = \frac{x}{6}, \varsigma x = 0, \vartheta x = x$$

From above it is clear that $\varpi(\Delta) \subseteq \sigma(\Delta), \varsigma(\Delta) \subseteq \vartheta(\Delta)$. *Define two sequences* $\{x_n\}$

and $\{y_n\}$ *in* Δ *such that*

$$x_n = \frac{1}{n} \quad and \quad y_n = \frac{2}{n}$$

Then clearly, $y_n \preceq x_n$ *and* $y_n \to 0 \preceq x_n$.

Also,

$$\lim_{n\to\infty} \varpi x_n = \lim_{n\to\infty} \frac{1}{12n} = 0 \in \Delta \quad and \quad \lim_{n\to\infty} \vartheta x_n = \lim_{n\to\infty} \frac{1}{n} = 0 \in \Delta$$

Hence $\lim_{n\to\infty} \varpi x_{n\to\infty} = \lim_{n\to\infty} \vartheta x_{n\to\infty} = 0(=q)$.

Thus the pair (ϖ,ϑ) *satisfied the* $b-$ *(E.A.) property.*

Similarly, one can see that the pair (ς,σ) *will also satisfy the* $b-$ $(E.A.)$ *property.*

Further, define the maps $\psi,\varphi : [0,\infty) \to [0,\infty)$ *as*

$$\psi(t) = \frac{7t}{8} \quad and \quad \varphi(t) = \frac{t}{8} \quad \forall\ t \in [0,\infty)$$

Without loss of generality, we can set $\varepsilon = 2$ *and* $s = 2$*, then from inequality* (1) *for all* $x, y \in \Delta$*, we have*

$$\begin{aligned} \text{L.H.S.} &= \psi\left(s^{\varepsilon} d(\varpi x, \varsigma y)\right) = \psi\left(s^{\varepsilon}|\varpi x - \varsigma y|^2\right) \\ \text{R.H.S.} &= \psi(M(x,y)) - \varphi(M(x,y)) \end{aligned}$$

where

$$\begin{aligned} M(x,y) &= \max\left\{ \begin{matrix} d(\vartheta x,\sigma y), d(\varpi x,\vartheta x), d(\varsigma y,\sigma y), \\ \frac{1}{2s}[d(\varpi x,\sigma y)+d(\vartheta x,\varsigma y)], \frac{1}{2}[d(\vartheta x,\sigma y)+d(\varpi x,\vartheta x)] \end{matrix} \right\} \\ &= \max\left\{ \tfrac{|6x-y|^2}{36}, \tfrac{121x^2}{144}, \tfrac{y^2}{36}, \tfrac{1}{2s}\left[\tfrac{|6x-12y|^2}{144} + x^2\right], \tfrac{1}{2}\left[\tfrac{|6x-y|^2}{36} + \tfrac{121x^2}{144}\right] \right\} \end{aligned}$$

For every $x > y$*, we have*

$$\text{L.H.S.} = \psi\left(s^{\varepsilon}|\varpi x - \varsigma y|^2\right) = \psi\left(s^{\varepsilon}\left|\frac{x}{12} - 0\right|^2\right) = \frac{7x^2}{36\times 8}$$

$$\begin{aligned} M(x,y) &= \frac{|6x-y|^2}{36} \\ \text{R.H.S.} &= \psi(M(x,y)) - \varphi(M(x,y)) \\ &= \psi\left(\frac{|6x-y|^2}{36}\right) - \varphi\left(\frac{|6x-y|^2}{36}\right) \\ &= \frac{|6x-y|^2}{36} \geq \frac{7x^2}{36\times 8} = \text{L.H.S.} \end{aligned}$$

Hence L.H.S. $\leq$ *R.H.S for all* $x, y \in \Delta$ *with* $x > y$. *Thus all the conditions of Theorem* (3.1) *are satisfied. Moreover,* 0 *is the unique common fixed point of the maps.*

4. Conclusion. In this paper, a fixed point theorems for two pairs of self maps satisfying (ψ, ϕ)-weak contractive with b-(E.A)-property in partially ordered b-metric spaces are proved. Our results extended some of the existing results of the literature such as the results of Ozturk & Radenovic (2015). One illustration has been provided to support our finding.

5. Future Scope. In this research, four mappings in partially ordered $b-$ metric space under various assumptions were used to develop a fixed theorem. The existence of a solution to a theoretical or real-world problem is analogous to the existence of a fixed point for an appropriate mapping or functions in a wide variety of computer, mathematical, economic, modeling, and engineering challenges. Interested authors can currently expand our research towards mathematical models of diseases.

References

Aamri, M. and **El Moutawakil, D.** (2022) : Some new common fixed point theorems under strict contractive conditions, *J. Math. Anal. Appl.*, **270**, 181–188.

Aghajani, A.; Abbas, M. and **Roshan, J.R.** (2014) : Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces, *Math. Slovaca*, **64(4)**, 941–960.

Alber, Y.I. and **Guerre-Delabriere, S.** (1997) : Principle of weakly contractive maps in Hilbert spaces, In: Gohberg, I., Lybich, Y. (eds.) *New Results in Operator Theory and Its Applications, OTAA*, **98**, 7–12.

Bakhtin, I.A. (1989) : The contraction mapping principle in almost metric spaces, *J. Funct. Anal.*, **30**, 26–37.

Banach, S. (1922) : Sur les opÃĪrations dans les ensembles abstraits et leur application aux ÃĪquations intÃĪgrales, *Fundam. Math.*, **3**, 133–181.

Bourbaki, N. (1974) : Topologie Generale, *Herman*, Paris, France.

Czerwik, S. (1993) : Contraction mapping in b - metric spaces, *Acta Math. Inform. Univ. Ostraviensis*, **1(1),** 5–11.

Czerwik, S. (1998) : Non linear set valued mapping in b - metric spaces, *Atte Sem. Univ. Modena*, **46**, 263–276.

Delbosco, D. (1976–77) : Un' estensione di un teorema sul punti fisso di S. Reich, *Rend. Sem. Mat. Univ. Torino*, **35**, 233–238.

Dulta, P.N. and **Choudhurry, B.S.** (2008) : A generaliation of contraction principle in metric spaces, *Fixed Point Theory Appl.*, **2008**, Article id. 406368.

Frechet, M. (1906) : Sur quelques points duo calcul fonctionel, *Rendicouti Mahematicodi Palermo*, **22**, 1–74.

Gupta, V.; Jungck, G. and **Mani, N.** (2020) : Some novel fixed point theorems in partially ordered metric spaces, *AIMS Mathematics*, **5(5)**, 4444–4452.

Gupta, V.; Mani, N. and **Tripathi, A.K.** (2012) : A fixed point theorem satisfying a generalized weak contractive condition of integral type, *International Journal of Mathematical Analysis*, **6**, 1883–1889.

Gupta, V. and **Mani, N.** (2013) : Existence and uniqueness of fixed point for contractive mapping of integral type, *Int. J. Computing Science and Mathematics*, **4(1)**, 72–83.

Gupta, V.; Ramandeep; Mani, N. and **Tripathi, A.K.** (2016) : Some fixed point result involving generalized altering distance function, *Procedia Computer Science*, **79**, 112–117.

Gupta, V.; Shatanawi, W. and **Mani, N.** (2017) : Fixed point theorems for (ψ, β)- GeraghtyâĂŹs contraction type maps in ordered metric spaces and some applications to integral and ordinary differential equations, *J. Fixed Point Theory Appl.*, **19(2)**, 1251–1267.

Jungck, G. (1976) : Commuting mappings and fixed points, *Amer. Math. Monthly*, **83**, 261–263.

Jungck, G. (1986) : Compatible mappings and common fixed points, *Int. J. Math. Math. Sci.*, **9**, 771–779.

Jungck, G. (1996) : Common fixed points for noncontinuous non-self maps on non-metric spaces, *Far East Journal of Mathematical Sciences*, **4(2)**, 199–215.

Kannan, R. (1968) : Some results on fixed points, *Bulletin of the Calcutta Mathematical Society*, **60**, 71–76.

Khan, M.S.; Swalech, M. and **Sessa, S.** (1984) : Fixed point theorems by altering distances between the points, *Bull. Aust. Math. Soc.*, **30**, 323–326.

Mani, N. (2018) : Generalized C_{β}^{ψ}-rational contraction and fixed point theorem with application to second order deferential equation, *Mathematica Moravica*, **22(1)**, 43–54.

Mitiku, B.; Karusala, K. and **Namana, S.R.** (2020) : Some fixed point results of generalized (ϕ, ψ) - contractive mappings in ordered b metric spaces, *BMC. Res. Notes*, **13**, 537.

Morales, J.R.; Rojas, E.M. and **Bisht, R.K.** (2014) : Common fixed points for pairs of mappings with variable contractive parameters, *Abstr. Appl. Anal.*, **2014**, Article id.7.

Morales, J.R. and **Rojas, E.M.** (2016) : Contractive mappings of rational type controlled by minimal requirements functions, *Afr. Mat.*, **27(1-2)**, 65–77.

Morales, J.R. and **Rojas, E.M.** (2021) : Common fixed points for $(\psi - \phi)$ - weak contractions type in $b-$ metric spaces, *Arab. J. Math.*, **10**, 639–658.

Nieto, J.J. and **Lopez, R.R.** (2005) : Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, *Order*, **22**, 223–239.

Nieto, J.J. and **Lopez, R.R.** (2007) : Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations, *Acta. Math. Sinica.*, **23**, 2205–2212.

Ozturk, V. and **Turkoglu, D.** (2015) : Common fixed point theorems for mappings satisfying (E.A)-property in b-metric spaces, *J Nonlinear Sci Appl*, **8**, 1127–1133.

Ozturk, V. and **Turkoglu, D.** (2015) : Fixed points for generalized $\alpha - \psi$-contractions in b-metric spaces, *J Nonlinear Convex Anal.*, **16(10)**, 2059–2066.

Pant, R.P. (1996) : Common fixed points of noncommutating mappings, *J. Math. Anal. Appl.*, **188**, 436–440

Ran, A.C.M. and **Reurring, M.C.B.** (2004) : Fixed point theorems, *Proceedings of the American Mathematical Society*, **132**, 1435–1443.

Rhoades, B.E. (2001) : Some theorems on weakly contractive maps, *Nonlinear Anal.*, **47(4)**, 26832693.

Sessa, S. (1982) : On a weak commutativity condition of mappings in fixed point considerations, *Publications de lnstitut Mathematique*, **32**, 149–153.

Skof, F. (1977) : Teorema di punti fisso per applicazioni negli spazi metrici, *Atti. Accad. Aci. Torino*, **111**, 323–329.

[1]DEPARTMENT OF MATHEMATICS
SANDIP UNIVERSITY
NASHIK, MAHARASHTRA, INDIA

E-MAIL : meghakothawade17@gmail.com

[2]DEPARTMENT OF MATHEMATICS
SANDIP UNIVERSITY
NASHIK, MAHARASHTRA, INDIA

E-MAIL : renupathak380@gmail.com

[3]DEPARTMENT OF MATHEMATICS
CHANDIGARH UNIVERSITY
PUNJAB, INDIA

E-MAIL : naveenmani81@gmail.com