Steiner tree problems

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1 Introduction

The Steiner tree is an NP-hard combinatorial optimization problem [50] with a long history[11, 93, 66]. The study of Steiner trees received great attentions in 1990s since many important open problems, including Gilbert-Polak conjecture on the Euclidean Steiner ratio, the existence of better approximation, and the existence of polynomial-time approximation schemes (PTAS), have been solved with influence in the general theory of designs and analysis of approximation algorithms for combinatorial optimizations, and also, many new important applications in VLSI designs, optical networks, wireless communications, etc. have been discovered and studied extensively. Those applications usually require some modifications on classical Steiner tree problems and hence require new techniques for solving them. Therefore, studying various variations of Steiner trees forms a hot point recently. In this article, we will review important developments in 1990s and discuss some open problems which may induce important developments in this centrary.

2 On the Proof of Gilbert-Pollak's Conjecture

Given a set of points in a metric space, the problem is finding a shortest network interconnecting the points in the set. Such a shortest network is called a *Steiner*

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minimum tree on the point set. The Steiner tree problem can be seen as a generalization of Fermat's problem. Three hundred years ago, Fermat proposed a problem of finding a point to minimize the total distance from this points to three given points in the Euclidean plane; its solution is exactly the Steiner minimum tree on the three points. The general form of Steiner minimum tree problem was proposed by Gauss [26]. However, Courant and Robbins [27] in their famous 1941 book " What is Mathematics" referred to it as the Steiner problem. The popularity of their book was responsible for bringing the Steiner tree problem to people's attention. Two important papers in the 1960's further laid a solid groundwork for further study. Melzak [75] first gave a finite algorithm for the euclidean Steiner trees. Gilbert and Pollak [52] produced an excellent survey of the problem, raised many new topics including Steiner ratio problem, and extended the problem to other metric space. Since then, more than three hundred research papers have been contributed to the Steiner tree problem. For an excellent survey, one may refer to the book of Hwang, Richards, and Winter [55].

An important developments on the Steiner tree problem that took place in the beginning of 1990s is the proof of Gilbert-Pollak's conjecture on the Euclidean Steiner ratio [34, 35]. This new development is based on a discovery of new approach with a new minimax theorem.

A *minimum spanning tree* on a set of points is the shortest network interconnecting the points in the set with all edges between the points. While the Steiner tree problem is intractable, the minimum spanning tree can be computed pretty fast. The Steiner ratio in a metric space is the largest lower bound for the ratio between lengths of a minimum Steiner tree and a minimum spanning tree for the same set of points in the metric space, which is a measure of performance for the minimum spanning tree as a polynomial-time approximation of the minimum Steiner tree. Determining the Steiner ratio in each metric space is a traditional problem on Steiner trees. In 1976, Hwang [54] determined that the Steiner ratio in rectilinear plane is 2/3. However, it took 22 years for completing the story of determining the Steiner ratio in the Euclidean plane. In 1968, Gilbert and Pollak conjectured that the Steiner ratio in the euclidean plane is $\sqrt{3}/2$. Through efforts made by Pollak [80], Du, Hwang and Yao [31, 32], Friedel and Widmayer[46], Booth [14], Rubinstein and Thomas [86, 87], Graham and Hwang [53], Chung and Hwang [24], Du and Hwang [31], and Chung and Graham [23], the conjecture was finally proved by Du and Hwang [34, 35] in 1990. The significance of their proof stems also from the potential applications of the new approach included in the proof.

In their approach, the central part is a new minimax theorem about minimizing the maximum value of several concave functions over a simplex as follows.

Theorem 1 (Du-Hwang Minimax Theorem) *Let $f(x) = \max_{i \in I} g_i(x)$ where I is a finite set and $g_i(x)$ is a continuous, concave function in a polytope X. Then the minimum value of $f(x)$ over the polytope X is achieved at some critical point, namely, a point satisfying the following property:*

(*) There exists an extreme subset Y of X such that $x \in Y$ and the index set $M(x)$ $(= \{i \mid f(x) = g_i(x)\})$ is maximal over Y.

The Steiner ratio problem is first transfered to such a minimax problem ($g_i(x)$= (the length of a Steiner tree) - (the Steiner ratio)·(the length of a spanning tree with graph structure i) where x is a vector whose components are edge-lengths of the Steiner tree) and the minimax theorem reduces the minimax problem to the problem of finding the minimax value of the concave functions at critical points. Then each critical point is transfered back to an input set of points with special geometric structure; it is a subset of a lattice formed by equilateral triangles. This special structure enables us to verify the conjecture corresponding to the non-negativeness of minimax value of the concave functions.

Clearly, in order to use the minimax approach, for each problem three questions will be addressed:

(1) How do we transfer the problem to such a minimax problem meeting the condition that the functions are concave?

(2) How do we determine the critical geometric structure?

(3) How do we verify the function value on the critical structure?

Developping techniques for answering these three questions will enable us to solve more open problems. Let us explain it by some examples in the following.

2.1 Chung-Gilbert's Conjecture

Steiner trees in Euclidean spaces have an application in constructing phylogenetic trees [17]. It was also conjectured by Gilbert and Pollak [52] that in any Euclidean space the Steiner ratio is achieved by the vertex set of a regular simplex. Chung and Gilbert [22] constructed a sequence of Steiner trees on regular simplices. The lengths of constructed Steiner trees goes decreasingly to $\sqrt{3}/(4-\sqrt{2})$. Although the constructed trees are not known to be Steiner minimum trees, Chung and Gilbert conjectured that $\sqrt{3}/(4-\sqrt{2})$ is the best lower bound for Steiner ratios in Euclidean spaces. Clearly, if $\sqrt{3}/(4-\sqrt{2})$ is the limiting Steiner ratio in d-dimensional Euclidean space as d goes to infinity, then Chung-Gilbert's conjecture is a corollary of Gilbert and Pollak's general conjecture. However, this general conjecture of Gilbert and Pollak has been disproved by Smith [92] for dimension from three to nine and by Du and Smith for dimension larger than two. Now, interesting questions which arise in this situation are about Chung and Gilbert's conjecture. Could Chung-Gilbert's conjecture also be false? If the conjecture is not false, can we prove it by the minimax approach?

First, we claim that Chung-Gilbert's conjecture could be true. In fact, we could get rid of Gilbert-Pollak's general conjecture, and use another way to reach the conclusion that the limiting Steiner ratio for regular simplex is the best lower bound for Steiner ratios in Euclidean spaces. To support our viewpoint, let us analyze a possible proof of such a conclusion as follows.

Consider n points in $(n-1)$-dimensional Euclidean space. Then all of $n(n-1)/2$ distances between the n points are independent. Suppose that we could do a similar transformation and the minimax theorem could apply to these n points to obtain a similar result in the proof of Gilbert-Pollak's conjecture for Euclidean plane, i.e. a point set with critical geometric structure has the property that the union of all minimum spanning trees contains as many equilateral triangles as possible. Then such a critical structure must be a regular simplex.

The above observation tells us two facts:

(a) Chung-Gilbert's conjecture can follow from the following two conjectures.

Conjecture 1 *The Steiner ratio for n points in an euclidean space is not smaller than the Steiner ratio for the vertex set of $(n-1)$-dimensional regular simplex.*

Conjecture 2 (Smith [92]) $\sqrt{3}/(4-\sqrt{2})$ *is the limiting Steiner ratio for simplex.*

(b) It may be possible to prove Conjecture 1 by the minimax approach if we could find a right transformation.

One may wonder why we need to find a right transformation. What happens to the transformation used in proof of Gilbert-Pollak's conjecture in the Euclidean plane? Here, we remark that such a transformation does not work for Conjecture 1. In fact, in the Euclidean plane, with a fixed graph structure, all edge-lengths of a full Steiner tree can determine the set of original points and furthermore the length of a spanning tree for a fixed graph structure is a convex function of the edges-lengths of the Steiner tree. However, in Euclidean spaces of dimension more than two , edge-lengths of a full Steiner tree are not enough to determine the set of original points. Moreover, adding other parameters may destroy the convexity of the length of a spanning tree as a function of the parameters.

Smith [92] showed by an exhaustive computation that for $d = 3, \cdots, 7$, the Steiner trees constructed by Chung and Gilbert are actually minimum Steiner trees, but, for $d = 8$, their Steiner tree is not minimum. He also conjectured that the trees of Chung and Gilbert are minimum if d is of the form $d = 3 \cdot 2^p$. Conjecture 2 is a corollary of this more specific conjecture.

¿From the above, we see that proving Chung-Gilbert's conjecture requires a further develoment of the minimax appoach.

2.2 Graham-Hwang's Conjecture

A Steiner tree with rectilinear distance is called a *rectilinear Steiner tree*. While rectilinear Steiner trees in plane have many applications on CAT and VLSI, rectilinear Steiner trees in high dimensional space can be found in biology [17, 48] and optimal traffic multicasting for some communication networks [13, 19]. Although the Steiner ratio in rectilinear plane was determined by Hwang [54] in earlier stage of the study of Steiner trees, there is still no progress on the Steiner ratio in rectilinear spaces by now. The Steiner ratio in a d-dimensional rectilinear space was conjectured to be $d/(2d-1)$ by Graham and Hwang [53]. The difficulty for extending Hwang's approach to proving Graham-Hwang's conjecture is due to the lack of knowledge on the full rectilinear Steiner trees in high dimensional spaces. (A full Steiner tree has a property that all original points are leaves.) In fact, for a full rectilinear Steiner tree in plane, all Steiner points lie on a path. However, it is not known whether a similar result holds for full rectilinear Steiner trees in a space of dimension more than two.

Graham-Hwang's conjecture can be easily transfered to a minimax problem requested by our minimax approach. For example, choose lengths of all straight segments of a Steiner tree. When connection pattern of the Steiner tree is fixed, the set of original points can be determined by such segments-lengths, the length of the Steiner tree is a linear function and the length of a spanning tree is a convex function of such segment-lengths, so that g_i is a concave function of such segment-lengths. However, for this transformation, it is hard to determine the critical structure. To explain the difficulty, we notice that in general the critical points could exist in both the boundary and interior of the polytopy. (See the minimax theorem.) In the proof of Gilbert-Pollak's conjecture in plane, a crutial fact is that only interior critical points need to be considered in a contradiction argument. The critical structure of interior critical points are relatively easy to be determined. However, for the current transformation on Graham-Hwang's conjecture, we have to consider some critical points on the boundary. It requires a new technique, either determine critical structure for such critical points or eliminate them from our consideration.

One possible idea is to combine the minimax approach and Hwang's method. In fact, by the minimax approach, we may get useful condition on the set of original points. With such a condition, the point set can have only certain type of full Steiner trees. This may reduce the difficulty of extending Hwang's method to high dimension.

The significance of developping techniques for determining critical structure corresponding to critical points on the boundary is not only for solving Graham-Hwang's conjecture, but also for solving some other problems. For example, it can be immediately applied to some packing problems. One of typical packing problems is to find the maximum number of objects which can be put in a certain container. When the objects are discs or spheres, the problem can be transfered to a minimax problem that meets our requirement. To determine such a number exactly, we have also to deal with critical points on the boundary of the polytope.

2.3 The Steiner Ratio in Banach Spaces

Examining the proof of Gilbert-Pollak's conjecture in Euclidean plane, we observe that the proof has nothing concerning the property of Euclidean norm except the last part, verification of the conjecture on point sets of critical structure. This means that using the minimax approach to determine the Steiner ratio in Minkowski plane (2-dimensional Banach space), we would have no problem on finding a transformation and determining critical structures. We would meet only a problem on verification for point sets with critical structure.

Steiner minimum trees in Minkowski planes have been studied by [1, 25, 70, 39, 90, 33]. In these papers, some fundamental properties of Steiner minimum trees in Minkowski planes have been established. Two nice conjectures about the Steiner ratio in Minkowski planes were proposed respectively by [25, 39] and [39] as follows:

Conjecture 3 *In any Minkowski plane, the Steiner ratio is between 2/3 and* $\sqrt{3}/2$.

Conjecture 4 *The Steiner ratio in a Minkowski plane equals that in its dual plane.*

With new techniques in the critical structures, Gao, Du, and Graham [49] proved the first half of Conjecture 3 that in any Minkowski plane, the Steiner ratio is at least 2/3, and Wan, Du, and Graham [97] showed that Conjecture 4 is true for three, four, and five points. With a different approach, Du et al [39] also proved that in any Minkowski plane, the Steiner ratio is at most 0.8766.

Chung-Gilbert conjectur and conjecture 4 can be extended to high-dimensional Banach spaces as follows.

Conjecture 5 *In any infinite dimensional Banach space, the Steiner ratio is between* $1/2$ *and* $\sqrt{3}/(2-\sqrt{2})$.

Conjecture 6 *The Steiner ratio in any Banach space equals that in its dual space.*

Significant results on these two conjectures could be produced by further developments of inimax approach from successful application in two-dimensional problems to high-dimension.

3 On Better Approximations

Starting from a minimum spanning tree, improve it by adding Steiner points. This is a natural idea to obtain an approximation solution for the Steiner minimum tree. Every approximation solution obtained in this way would have a performance ratio at most the inverse of the Steiner ratio. The problem is how much better than the inverse of the Steiner ratio one can make.

Over more than twenty years numerous heuristics [6, 13, 18, 44, 61, 63, 64, 65, 67, 94, 100] for Steiner minimum trees have been proposed for points in various mertic spaces. Their superiority over minimum spanning trees were often claimed by computation expierements. But no theoretical proof of superiority was ever given. It was a long-standing problem whether there exists a polynomial-time approximation with performance ratio better than the inverse of the Steiner ratio or not. For simplicity, a polynomial-time approximation with performance ratio smaller than the inverse of the Steiner ratio will be called a *better* approximation. The first significant work on better approximations was made by Bern [10]. He proved that for the rectilinear metric and Poisson distributed regular points, a greedy approximation obtained by a very simple improvement over a minimum spanning tree has a shorter average length. Later, Hwang and Yao [56] extended this result to the usual case when the number of regular points is fixed.

In 1991, Zelikovsky [101] made the first breakthrough to the problem by giving a better heuristic for the Steiner minimum trees in graph. This is the second important development on Steiner trees in 1990s. To explain his idea and review further development from his work, let us start from comparing his work with a previous work with a similar idea.

3.1 Chang's Idea

Chang [18, 19] proposed the following approximation algorithm for Steiner minimum trees in the Euclidean plane: Start from a minimum spanning tree and at each iteration choose a Steiner point such that using this Steiner point to connect three vertices in the current tree could replace two edges in the minimum spanning tree and this replacement achieves the maximum saving among such possible replacements.

Smith, Lee, and Liebman [91] also use the idea of the greedy improvement. But, they start with Delaunay triangulation instead of a minimum spanning tree. Since every minimum spanning tree is contained in Delaunay triangulation, the performance ratio of their approximation algorithm can also be bounded by the inverse of the Steiner ratio. The advantage of Smith-Lee-Liebman algorithm is on the running time. While Chang's algorithm runs in $O(n^3)$ time, Smith-Leeh-

Liebman algorithm runs only in $O(n \log n)$ time.

Kahng and Robin [60] proposed an approximation algorithm for Steiner minimum trees in the rectilinear plane by using the same idea as that of Chang. For these three algorithms, it can be proved that for any particular set of points, the ratio of lengths of the approximation solution and the Steiner minimum tree is smaller than the inverse of the Steiner ratio. Some experiemental results also show that the approximation solution obtained by these algorithms are very good. However, no proof has been found to show any one of them being a better approximation.

3.2 Zelikovsky's Idea

Zelikovsky's idea [101] is based on the decomposition of a Steiner tree (namely, a tree, not necessarily minimum, interconnecting original points): An original point in a Steiner tree can be either a leaf or a junction. In the latter case, the Steiner tree can be decomposed at this point. In this way, every Steiner tree can be decomposed into edge-disjoint union of several Steiner trees for subsets of original points; each of them has no junction being an original point. A Steiner tree with no original point being a junction is called a *full* Steiner tree. The full Steiner trees in the decomposition are called *full components*. The size of a full component is the number of original points in the component.

Clearly, for any $k \geq 3$, a k-size Steiner minimum tree usually has shorter length compared with a minimum spanning tree. It is natural to think about using a minimum k-size Steiner tree to approximate the Steiner minimum tree. However, this does not work because computing a k-size Steiner minimum tree is still an intractable problem. Zelikovsky's idea is to approximate the Steiner minimum tree by a 3-size Steiner tree generated by a polynomial-time greedy algorithm. The key fact is that the length of such a heuristic is smaller than the arithmetic mean of lengths of a minimum spanning tree and a 3-size Steiner minimum tree; that is, the performance ratio of his approximation satisfies

$$PR \leq \frac{\rho_2^{-1} + \rho_3^{-1}}{2}$$

where ρ_k is the k-Steiner ratio. Thus, if the 3-Steiner ratio ρ_3 is bigger than the Steiner ratio ρ_2, then this greedy algorithm is a better approximation for the Steiner minimum tree. Zelikovsky was able to prove that 3-Steiner ratio in graphs is at least $3/5$ which is bigger than $1/2$, the Steiner ratio in graphs [61].

So, he solved the better approximation problem in graphs. Zelikovsky's idea has been extensively studied in the literature.

Du, Zhang, and Feng [36] generalized Zelikovsky's idea to the k-size Steiner tree. They showed that a generalized Zelikovsky's algorithm has performance ratio

$$PR \leq \frac{(k-2)\rho_2^{-1} + \rho_k^{-1}}{k-1}.$$

Berman and Ramaiyer [9] employed a different idea to generalize Zelikovsky's result. They obtained an algorithm with the performance ratio satisfying

$$PR \leq \rho_2^{-1} - \sum_{i=3}^{k} \frac{\rho_{i-1}^{-1} - \rho_i^{-1}}{i-1}.$$

They also showed that in the rectilinear plane, the 3-Steiner ratio is at least 72/94 which is bigger than 2/3 [54], the Steiner ratio in rectilinear plane. So, they solved the better heuristic problem in rectilinear plane.

Du, Zhang, and Feng [36] proved a lower bound for the k-Steiner ratio in any metric space. This lower bound goes to one as k goes to infinity. So, in any metric space with the Steiner ratio less than one, there exists a k-Steiner ratio bigger than the Steiner ratio. Thus, they proved that the better heuristic exists in any metric space satisfying the following conditions:

(1) The Steiner ratio is smaller than one.

(2) The Steiner minimum tree on any fixed number of points can be computed in polynomial-time.

These metric spaces include Euclidean plane and Euclidean spaces.

Zelikovsky [104] used a different potential function in his greedy approximation and obtained an approximation with performance ratio satisfying

$$PR \leq \rho_k^{-1}(1 - \ln \rho_2).$$

Although Zelikovsky's idea starts from a point different from Chang's one, the two approximations are actually similar. To see this, let us describe Zelikovsky's algorithm as follows: Start from a minimum spanning tree and at each iteration choose a Steiner point such that using this Steiner point to connect three regular points could replace two edges in the minimum spanning tree and this replacement achieves the maximum saving among such possible replacements.

Clearly, they both start from a minimum spanning tree and improve it step by step by using a greedy principal to choose a Steiner point to connect a triple of vertices. The difference is only that this triple in Chang's algorithm may contain some Steiner points while it contains only regular points in Zelikovsky's algorithm. This difference makes Chang's approximation hard to be analyzed. Which one will give a better approximation solution? This is an interesting problem.

3.3 The k-Steiner Ratio ρ_k

While the determination of the k-Steiner ratio plays an important role in estimation of the performance ratio of several recent better approximations, Borchers and Du [15] completely determined the k-Steiner ratio in graphs that for $k = 2^r + h \geq 2$,

$$\rho_k = \frac{r2^r + h}{(r+1)2^r + h}$$

and Borchers, Du, Gao, and Wan [16] completely determined the k-Steiner ratio in the rectilinear plane that $\rho_2 = 2/3$, $\rho_3 = 4/5$, and for $k \geq 4$, $\rho_k = (2k-1)/(2k)$. However, the k-Steiner ratio in the Euclidean plane for $k \geq 3$ is still an open problem. Du, Zhang, and Feng [36] conjectured that the 3-Steiner ratio in the Euclidean plane is

$$\frac{(1+\sqrt{3})\sqrt{2}}{1+\sqrt{2}+\sqrt{3}}.$$

They also analyzed that the k-Steiner ratio in the Euclidean plane might be determined in a similar way to the proof of Gilbert-Pollak conjecture. The difficulty appears only in the description of "critical structure".

3.4 Variable Metric Method

Berman and Ramaiyer [9] introduced an interesting approach to generalize Zelikovsky's greedy approximation. Let us call the Steiner minimum tree for a subset of k regular points as a *k-tree*. Their approach consists of two steps. The first step processes all i-trees, $3 \leq i \leq k$, sequentially in the following way: For each i-tree T with positive saving in the current graph, put T in a stack and if two leaves x and y of T are connected by a path p in a minimum spanning tree without passing any other leaf of T, then put an edge between x and y with weight equal to the length of the longest edge in p minus the saving of T. In the second step, it repeatedly pops i-trees from the stack remodifying the original minimum spanning tree for all regular points and keeping only i-trees with the current positive saving. Adding weighted edges to a point set would change the metric on the points set. Let E be an arbitrary set of weighted edges such that adding them to the input metric space makes all i-trees for $3 \leq i \leq k$ have non-positive saving in the resulting metric space M_E. Denote by $t_k(P)$ a supermum of the length of a minimum spanning tree for the point set P in metric space M_E over all such E's. Then Berman-Ramaiyer's algorithm produces a k-size Steiner tree with total length at most

$$t_2(P) - \sum_{i=3}^{k} \frac{t_{i-1}(P) - t_i(P)}{i-1} = \frac{t_2(P)}{2} + \sum_{i=3}^{k-1} \frac{t_i(P)}{(i-1)i} + \frac{t_k(P)}{k-1}.$$

The bound for the performance ratio of Berman-Ramaiyer's approximation in subsection 3.2 is obtained from this bound and the fact that $t_k(P) \leq \rho_k^{-1} SMT(P)$ where $SMT(P)$ is the length of the Steiner minimum tree for point set P.

Based on the above observation, we may have the following questions. Could we find other way to vary metric for a better bound? Could we forget the greedy idea and design a better approximation with only variable metric idea? Answering these questions requires deeper understanding the variable metric method. We attempt to obtain new algorithms from this study.

Karpinski and Zelikovsky [62] proposed a preprocessing procedure to improve existing better approximations. First, they use this procedure to choose some Steiner points and then run a better approximation algorithm on the union of the set of regular points and the set of chosen Steiner points. This preprocessing improves the performance ratio for every known better approximation that we mentioned previously.

The preprocessing procedure is similar to the algorithm of Berman and Ramaiyer. But, it uses a "related gain" instead of the saving as the greedy function. One of our current ideas is to modify Chang's algorithm in the following way: At each iteration, if a Steiner point is introduced, then computer its related gain, and later consider only triples of regular points and Steiner points with positive related gain. Would this approximation perform better? We attempt to get the answer.

Although many better approximations have been found in recent years, none of them has performance ratio smaller than the inverse of 3-Steiner ratio. The inverse of the 3-Steiner ratio seems to be the limit for the performance ratio of polynomial-time approximations for Steiner minimum trees to be able to reach.

Arora *et al.* [5] conjectured that their backtrack greedy technique gives a polynomial-time approximation scheme to 3-size Steiner minimum trees. If their conjecture is true, then their algorithms also give approximations for Steiner minimum trees with performance ratio approach to the inverse of the 3-Steiner ratio. This probably is the best possible performance ratio. Thus, the conjecture of Arora *et al.* is an attractive problem to our further research.

A more accurate analysis [104, 62, 102] for the performance ratios of Berman-Ramaiyer's algorithm and Karpinski-Zelikovsky's preprocessing requires bounds for t_k and a similar number t^k. The techniques in [15, 16] for determining the k-Steiner ratio seems very promising for establishing tight upper bounds for t_k and t^k.

The knowledge for the lower bound of the performance ratio is widely open. One knows only that for Steiner minimum trees in graphs, if NP $\neq$ P then a lower bound larger than one exists, because the problem in this case is MAX SNP-complete [12].

4 On PTAS

Jiang et al. [58, 59] brought a quite different idea from previous ones to Steiner minimum trees. They decompose the set of regular points based on the lengths of edges in a minimum spanning tree. By an interesting analysis, they proved that if the ratio of lengths between the longest edge and the shortest edge in a minimum spanning tree is bounded by a constant, then there is a polynomial-time approximation scheme (PTAS) for Steiner minimum trees in the rectilinear plane and in the Euclidean plane. This idea can also be used in other geometric optimization problems, in particular, some variations of Steiner tree problems described in the next section.

In 1995, S. Arora and J. Mitchell independently discovered powerful techniques to establish polynomial-time approximation schemes for geometric optimization problems, including Euclidean and rectilinear Steiner tree problems. Their results constitute the third important development on Steiner trees in 1990s. The significance of their results is not only on Steiner trees, but also on the design and analysis of aproximation algorithms in combinatorial optimization. Let us review these two remarkable techniques in the following.

4.1 Arora's PTAS

It is quite interesting to notice that Arora [3] appeared only one week before Mitchell [77]. Any way, they use very different techniques to reach the same goal. Therefore, both are very interesting. Arora's technique is based on recursive partition. In Jiang et al [58, 59], although partition can be moved parallelly, the size of each cell is fixed. It cannot be varied according to local information about distibution of terminals. Therefore, only in case that terminals are distributed almost evenly, the partition could work well. This is why such a condition that the ratio of lengths between the longest edge and the shortest edge in a minimum spanning tree is bounded by a constant is required.

However, in Arora's recursive partition, each big cell is partitioned into small cells independently from other big cells. How to cut only depends on the situation inside of itself. This advantage enables him to discard the condition in Jiang et al [58, 59].

4.2 Mitchell's PTAS

Mitchell's technique was initiated from studying a minimum length rectangular partition problem. Given a rectilinear region R surrounded by a rectilinear polygon and some rectilinear holes, a *rectangular partition* of R is a set of segments in R, which divide R into small rectangles each of which does not contain any hole in its interior. The problem is to find such a rectangular partition with the minimum total length. This problem is NP-hard.

Du et al. [29] introduced a concept of guillotine subdivision. A guillotine subdivision is a sequence of cuts performed recursively such that each cut partitions a piece ito at least two. Du et al. [29] showed that the minimum length guillotine rectangular partition can be computed in polynomial-time. However, they were only able to show that this guillotine subdivision is an approximation of the minimum length rectangular partition problem with performance ratio two in a special case that the region R is surrounded by a rectangle with some points as holes in it. Mitchell [76] showed that this is actually true in general. He also successfully utilized this technique to obtain constant approximations for other geometric optimization problems. With the same technique, Mata

[74] obtained a constant-factor approximation algorithm for red-blue separation problem improving previous result $O(logn)$.

Inspired by this success, Mitchell [77] extended guillotine subdivision to m-guillotine subdivision, a rectangular polygonal subdivision such that there exists a cut whose intersection with the subdivision edges consists of a small number ($O(m)$) of connected components and the subdivisions on either side of the cut are also m-guillotine. With a minor change of the proof of [76], Mitchell established a PTAS for minimum length rectangular partition problem. Mitchell [78, 79] further extended this m-guillotine subdivision technique to other geometric optimization problems, including Euclidean and rectlinear Steiner tree problems, and obtained PTAS for them.

5 Variations of Steiner Trees

Successful researches on classical Steiner tree problems encourage extensive study on variations of Steiner trees with various application backgrounds. Currently, they form a quite active research direction in Steiner trees.

In VLSI design, one considers several sets of terminals and finds a minimum total length packing of Steiner trees for these sets under the following situation [82]: The edges of the Steiner trees are required to lie in channels between cells. Each channel has a capacity which tells at most how many edges can run through it.

A complicated computer network usually consists of several nets of different speeds. The following problem was proposed based on such a background: Consider an undirected network with multiple edge weights $(c_1(e), c_2(e), ..., c_k(e))$ $(c_1(e) > c_2(e) > \cdots > c_k(e))$. Given a subset N of vertices and a partition $\{N_1, N_2, ..., N_k\}$ of N with $|N_1| \geq 2$, find a subnetwork interconnecting N with minimum total weight such that the length of any edge e on a path between a pair of vertices in N_j is at least $c_j(e)$ [57, 43].

To construct roads of minimum total length to interconnect n highways under the constraint that the roads can intersect each highway only at one point in a designated interval which is a line-segment, a generalization of Euclidean Steiner trees has been proposed and studied. Du, Hwang, and Xue [40] presented a set of optimality conditions for the problem and showed how to construct a solution to meet this set of optimality conditions.

Constructing phylogenetic trees is an important topic in computer biology. One of formulations is as follows: For a fixed alphabet A, let d denote the Hamming distance on A^n, i.e. $d((a_1, \cdots, a_n), (b_1, \cdots, b_n))$ equals the number of indices i such that $a_i \neq b_i$. Given a set P of points in the metric space (A^n, d), find a Steiner minimum tree for P. This problem is known to be NP-hard. (See [48].)

When a new customer is out of original telephone network, the company has to build a new line to connect the customer into the network. This situation brings us an on-line Steiner tree problem as follow: Assume that a sequence of points in a metric space are given step by step. In the ith step, only locations of the first n_i points in the sequence are known. The problem is to construct a shorter network at each step based on the network constructed in previous steps. The study of on-line problems was initiated from Sleator and Tarjan [89] and Manase, McGeoch, and Sleator [73]. A criterion for the performance of an on-line algorithm is to compare the solution generated by the on-line algorithm with the solution of corresponding off-line problem. In the Euclidean plane, it has been known that the worest-case ratio of lengths between on-line solution and off-line solution is between $O(n \log n / \log \log n)$ and $O(n \log n)$ [2, 96, 99].

Listing all variations and review each of them may take tremendous time and space. It should not be the job of this shorter article. Therefore, we next review a few for which some significant results are obtained recently.

5.1 Steiner Arborescence

Given a weighted directed graph G, a vertex r, and a subset P of n vertices, a *Steiner arborescence* is a directed tree with root r such that for each $x \in P$ there exists a path from r to x. The shortest Steiner arborescence is also called a *minimum Steiner arborescence*. Computing minimum Steiner arborescence is an NP-hard problem. Also, one knows that if NP $\neq$ P, then the best possible performance ratio of polynomial-time approximation for this problem is $O(\log n)$. This means that although, like the minimim spanning tree, the minimum arborescence as a shortest arborescence tree without Steiner points can be computed in polynomial-time, the Steiner ratio (the maximum lower bound for the ratio of lengths between the minimum Steiner arborescence and the minimum arborescence for the same set of given points) in directed graphs is zero. Dai et al [28, 20] apply Arora's techniques to this problem and obtained the best known result that for any $\varepsilon > 0$ there exists a polynomial-time approximation with performance ratio $O(n^{\varepsilon})$. An open problem reamins for closing the gap between the lower bound and the upper bound for the performance ratio.

A version of this problem in the rectilinear plane has a great interest in VLSI designs and an interesting story in the literature. Given a set P of n points in the first quadrant of the rectilinear plane, a *rectilinear Steiner arborescence tree* is a directed tree rooted at the origin, consisting of all paths from the root to points in P with horizontal edges oriented in left-to-right direction and vertical edges oriented in bottom-up direction. What is the complexity of computing the minimum rectilinear arborescence? First, it was claimed that a polynomial-time algorithm was found. However, Rao, Sadayappan, Hwang, and Shor [84] found a serious flow in this algorithm. Although they could not show the NP-completeness of the problem, they pointed out the difficulties of computing the minimum rectilinear arborescence in polynomial-time. They also showed that while the ratio of lengths between a minimum arborescence tree and a minimum Steiner tree for the same set of points tends to infinity, there is a polynomial-time approximation with performance two. Recently, Shi and Su [88] showed that computing the minimum rectilinear arborecence is NP-hard. Lu and Ruan [72] showed, by employing Arora's techniques, that there is a polynomial-time approximation scheme for the problem.

5.2 Edge-length and Number of Steiner Points

In wavelength-division multiplexing (WDM) optical network design [68, 83], suppose we need to connect n sites located at $p_1, p_2, \cdots, p_n$ with WDM optical network. Due to the limit in transmission power, signals can only travel a limited distance (say R) for guaranteed correct transmission. If some of the inter-site distances are greater than R, we need to provide some amplifiers or receivers/transmitters at some locations in order to break it into shorter pieces. This situation requires us to consider the problem of minimizing the maximum edge-length and the number of Steiner points in design of WDM optical network. To do so, two variations of Steiner trees have been studied.

The first is to minimize the number of Steiner points under upper bound for edge-length. That is, given a set of n terminals $X = \{p_1, p_2, \cdots, p_n\}$ in the Euclidean plane $\mathcal{R}^2$, and a positive constant R, the problem is to compute a tree T spanning a superset of X such that each edge in the tree has a length no more than R and with the minimum number $C(T)$ of points other than those in X, called *Steiner points*. This problem is called *Steiner tree problem with minimum number of Steiner points*, denoted by *STP-MSP* for short. Lin and Xue[69] showed that the STP-MSP problem is NP-hard. They also showed that the approximation obtained from the minimum spanning tree by simply breaking each edge into small pieces within the upper bound (called steinerized spanning tree) has a worst-case performance ratio at most five. Chen et al [21] showed that this approximation has a performance ratio exactly four. They also presented a new polynomial-time approximation with a performance ratio at most three and a polynomial-time approximation scheme under certain conditions. Lu et al [71] studied the STP-MSP in rectilinear plane. They showed that in the rectilinear plane, the steinerized spanning tree has performance ratio exactly three and there exists a polynomial-time approximation two.

The second is to minimize the maximum edge-length under an upper bound on the number of Steiner points. That is, given a set $P = \{p_1, p_2, \ldots, p_n\}$ of n terminals and an positive integer k, we want to find a Steiner tree with at most k Steiner points such that the length of the longest edges in the tree is minimized. This is one of the bottleneck Steiner tree problems. Wang and Du [98] showed that (a) if $NP \neq P$, then the performance ratio of any polynomial-time approximation for the problem in the Euclidean plane is at least $\sqrt{2}$; (b) if $NP \neq P$, then the performance ratio of any polynomial-time approximation for the problem in the rectilinear plane is at least two; (c) there exists a polynomial-time approximation with performance ratio two for the problem in both rectilinear and Euclidean planes.

5.3 Multiphase

Given an edge-weighed complete graph with vertex set X ($|X| = n$) and subsets $X_1, \cdots, X_m$ of vertices, the problem is to find a minimum weighed subgraph G such that for every $i = 1, \cdots, m$, G contains a spanning tree for X_i. This problem is called *subset interconnection designs* or *multiphase spanning network* problem [37, 38]. Du et al [**?**] showed that if NP $\neq$ P, then the best performance ratio of polynomial-time approximation for this problem is $\ln n + O(1)$.

Given an edge-weighed graph B with vertex set X and subsets $X_1, Y_1, \cdots, X_m, Y_m$ of X with $X_i \cap Y_i = \emptyset$, the problem is to find a minimum weighed subgraph G such that for every $i = 1, \cdots, m$, G contains a Steiner tree for X_i without using vertices not in Y_i. This problem is called *multiphase Steiner network* problem. Both multiphase spanning network and Steiner network problems arose in communication network design [81] and vacuum system design [38]. For the former one, when the solution is a forest, the system $(X_1, \cdots, X_m)$ is called *subtree hypergraph.* Such a system has various applications in computer database schemes [7] and statistics. It is also related to chordal graphs [42, 45]. Tarjan and Yannakakis [95] gave a $O(m+n)$-time algorithm to tell whether a set system is a subtree hypergraph or not.

Comparing the phylogenetic tree problem with multi-phase Steiner network problem, we would find some similarities between them if we look at each coordinate like a phase. For multi-phase Steiner tree problem, if the solution is a tree, then we have either a good heuristic or a polynomial-time computable exact solution[38]. This suggests that studying the relationship between the two problems will hopefully find a new construction of phylogenetic trees.

Ruan et al [85] found that multi-weight Steine tree problem can be transformed to multiphase Steiner tree problem. This initiates new line to study both problems.

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