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EXPONENTIAL DIOPHANTINE EQUATION IN THREE UNKNOWNS

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Abstract

In this manuscript, we demonstrate with the purpose of the given two singular Exponential Diophantine equation $E_1 : 23^x + 24^y = z^2$ and $E_2 : 62^x + 63^y = z^2$ has a unique solution in $N \cup \{0\}$. The solution (x, y, z) are $(0, 1, 5)$ and $(0, 1, 8)$ respectively.

I. Introduction

In 2007, Acu [1] proved that (3, 0, 3) and (2, 1, 3) are only two solutions (x, y, z) for the Diophantine Equation $2^x + 5^y = z^2$ where x, y and z are non-negative integers. In 2011, Suvarnamnai, Singta and Chotchaisthit [3] proved that the two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no non-negative integer solution. In 2012, Chotchaisthit [3] found all non-negative integer solutions for the Diophantine equation of type $4^x + p^y = z^2$ where p is a positive prime number.

In this paper, we show that with the purpose of the given two singular exponential Diophantine equation $E_1 : 23^x + 24^y = z^2$ and $E_2 : 62^x + 63^y = z^2$ has a unique solution in $N \cup \{0\}$. The solution (x, y, z) are $(0, 1, 5)$ and $(0, 1, 8)$ respectively.

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II. Preliminaries

In this part we make use of the factorizable technique and Catalan's conjecture to donate with indication the two lemmas.

Proposition 2.1 [4] (The Catalan's conjecture). (3, 2, 2, 3) *is a only one of its kind explanation* (a, b, x, y) *designed for the Diophantine equation* $E : a^x - b^y = 1$ *where* a, b, x *and y are integers with min* $\{a, b, x, y\} \geq 2$.

Lemma 2.2. *The Diophantine equation* $E_{11} : 23^x + 1 = z^2$ *has no non-negative integer points*.

Proof. Presume that at hand are non-negative integers *x* and *y* such that $23^x + 1 = z^2$. We deal with different cases.

Case 1: Fix, $x = 0$ implies $z^2 = 2$, which is contradiction to $z \in N \cup \{0\}$. We conclude that $x \geq 1$.

Case 2: Now $x \geq 1$, Thus, $E_{11} : z^2 = 23^x + 1 \geq 23^1 + 1 = 24$. Then $z \geq 5$. At the moment, we think about on the equation $E_{11} : z^2 - 23^x = 1$. By Proposition 2.1, E_{11} there is no integer points in $N \cup \{0\}$, if $x > 1$. Then we conclude that implies that $z^2 = 24$. This is a contradiction. Hence, the equation $23^x + 1 = z^2$ has no non-negative integer solution.

Lemma 2.3. (1, 5) *is a inimitable integer point* (y, z) *for the Diophantine Equation* $E_{12} : 1 + 24^y = z^2$ *everywhere y and z are non-negative integers*.

Proof. Suppose *y* and *z* are non-negative integers such with the intention of $1 + 24^y = z^2$. Now, we consider the equation E_{12} in different cases.

Case (i): Fix, $y = 0$ followed by $z^2 = 2$, which is unfeasible (*z* does not belongs to $N \cup \{0\}$).

Case (ii): Fix, $y \geq 1$. Therefore,

$$\begin{aligned} z^2 &= 24^y + 1 \\ &\geq 24 + 1 = 25. \end{aligned}$$

At that time, $z \geq 5$. At the moment, we think about on the equation $E_{12} : z^2 - 24y = 1$. By Proposition 2.1, E_{12} there is no integer points in $N \cup \{0\}$, if $y > 1$. Then we conclude that $y = 1$, implies that $z = 5$. Hence, (1, 5) is a unique points (y, z) for the equation $E_{12} : 1 + 24^y = z^2$ where y and z are non-negative integers.

By lemma 2.3, we can see that $23^0 + 24^1 = 5^2$. In the after that section, we determination establish the focal effect.

Lemma 2.4. *The Diophantine equation* $E_{21} : 62^x + 1 = z^2$ *has no non-negative integer points*.

Proof. Assume that there are non- negative integers x and z such that $62^x + 1 = z^2$. We will discuss in this satiation in several cases.

Case (i): Fix $x = 0$ implies the equation E_{21} gives $z^2 = 2$ which is without a solution in $N \cup \{0\}$.

Case (ii): Now, the value of $x \geq 1$. Thus the equation E_{21} implies $z^2 = 62^x + 1 \geq 62^1 + 1 = 63$.

That means $z \geq 8$. Now, we consider on the equation $E_{21} : z^2 - 62^x = 1$. By Proposition 2.1, if $x > 1$, there is no integer points other than $(z, x) = (3, 2)$, then we have $x = 1$. Then $z^2 = 63$. This is a contradiction to $z \in N \cup \{0\}$. Hence, the equation $62^x + 1 = z^2$ has no non-negative integer points.

Lemma 2.5. (1, 8) *is a single point of* (y, z) *for the equation* $E_{22} : 1 + 63^y = z^2$ *where y and z are non-negative integers*.

Proof. Let y and z be non-negative integers such that $1 + 63^y = z^2$. If $y = 0$, then $z^2 = 2$ which is impossible. Then $y \geq 1$. Thus, $z^2 = 63^y + 1 \geq 63 + 1 = 64$. Then, $z \geq 8$. Now, we consider on the equation $z^2 - 63^y = 1$. By Proposition 2.1, we have $y = 1$. Then, $z = 8$. Hence, (1, 8) is a unique solution (y, z) for the equation $1 + 63^y = z^2$ where y and z are non-negative integers.

By lemma 2.5, we can see that $62^0 + 63^1 = 8^2$. In the next section, we will present the most important consequence.

III. Main Results

Theorem 3.1. $(0, 1, 5)$ *is a single integral points* (x, y, z) *for the Diophantine equation* $E_{M1} : 23^x + 24^y = z^2$ *where* x, y *and* z *are non-negative integers.*

Proof. Let (x, y) and z be non-negative integers such that $23^x + 24^y = z^2$. By Lemma 2.2, we have $y \geq 1$. Now, we divide the number x into two cases.

Case (i): Fix, $x = 0$. By Lemma 2.3, we have only the integral point is $(y, z) = (1, 5)$.

Case(ii). Fix, $x \geq 1$. Make a note of that z is an odd integer. Then $z^2 \equiv 1 (\text{mod } 4)$. If x is odd, then $23^x \equiv 3 (\text{mod } 4)$, it follows that $24^y \equiv 2 (\text{mod } 4)$. This is a contradiction to $24^y \equiv 0 (\text{mod } 4)$. Hence conclude that x is only non-negative even number.

Suppose that, x is any non-negative even number. That is $x = 2k$, where $k \in N \cup \{0\}$. Then $z^2 - 23^{2k} = 24^y$. Using factorization methods to finding the solutions.

Case (i). Assume that, $(z + 23^k)(z - 23^k) = 24^y$ which equivalent is to $(z - 23^k) = 2^{2y}$ and $(z + 23^y) = 2^y(3^y)$. It follows that $2(23^k) = 2^y(3^y) - 2^{2y} = 2^y(3^y - 2^y)$ which implies $2 = 2^y$ and $3^y - 2^y = 23^k$, its only possible $y = 1$. It gives $3^y - 2^y = 23^0$. Hence $x = 0$, which is only suitable non-negative even integer.

Case (ii): Suppose that, $(z + 23^k)(z - 23^k) = 24^y$ which equivalent is to

$(z + 23^k) = 12^y$ and $(z - 23^k) = 2^y$. By the factoring that $2(23^k) = 2^y(6^y - 1)$ which implies $2^y = 2$ and $23^k = 6^y - 1$. It follows that $y = 1$ but $23^k \neq 5, \forall k \in N$.

Therefore, (0, 1, 5) is a single point (x, y, z) for the equation $23^x + 24^y = z^2$ where x, y and z are non-negative integers.

Corollary 3.2. *The Diophantine equation* $23^x + 24^y = w^4$ *has no non-negative integer point*.

Proof. Assume that there are non-negative integers x, y and w such that $23^x + 24^y = w^4$. Let $z = w^2$. Then $23^x + 24^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (0, 1, 5)$.

Then $w^2 = z = 5$. This is a contradiction to w is an integer. Hence, the equation $23^x + 24^y = w^4$ has no non-negative integer solution.

Theorem 3.3. (0, 1, 8) *is a unique solution* (x, y, z) *for the Diophantine equation* $E_{M2} : 62^x + 63^y = z^2$ *where* x, y *and* z *are non-negative integers*.

IV. Conclusion

We announcement in our consequences that 23 is odd prime number and $24 - 23 = 1$. Let p be an odd prime number. We possibly will request for the set of all points (x, y, z) for the Diophantine equation $E : p^x + (p+1)^y = z^2$ where x, y and z are non-negative integer.

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