Restricted inverse optimal value problem on linear programming under weighted l_1 norm

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Abstract

We study the restricted inverse optimal value problem on linear programming under weighted l_1 norm ($RIOVLP_1$). Given a linear programming problem LP_c : $\min\{cx|Ax = b, x \geq 0\}$ with a feasible solution x^0 and a value K, we aim to adjust the vector c to $\bar{c}$ such that x^0 becomes an optimal solution of the problem $LP_{\bar{c}}$ whose objective value $\bar{c}x^0$ equals K. The objective is to minimize the distance $\|\bar{c} - c\|_1 = \sum_{j=1}^{n} d_j|\bar{c}_j - c_j|$ under weighted l_1 norm. Firstly, we formulate the problem ($RIOVLP_1$) as a linear programming problem by dual theories. Secondly, we construct a sub-problem (D^z), which has the same form as LP_c, of the dual ($RIOVLP_1$) problem corresponding to a given value z. Thirdly, when the coefficient matrix A is unimodular, we design a binary search algorithm to calculate the critical value z^* corresponding to an optimal solution of the problem ($RIOVLP_1$). Finally, we solve the ($RIOV$) problems on Hitchcock and shortest path problem, respectively, in $O(T_{MCF} \log \max\{d_{max}, x^0_{max}, n\})$ time, where we solve a sub-problem (D^z) by minimum cost flow in T_{MCF} time in each iteration. The values d_{max}, x^0_{max} are the maximum values of d and x^0, respectively.

Keywords: Combinatorial optimization, Linear programming, Inverse optimization problem, Restricted inverse optimal value problem, l_1 norm

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1. Introduction

Since Burton and Toint (1992) first studied the inverse shortest path problem, many researchers have considered different inverse combinatorial optimization problems such as inverse spanning tree problem, inverse shortest path problem, inverse minimum cost flow problem, inverse minimum cut problem, inverse maximum matching problem. The inverse combinatorial optimization problems have broad applications which can be found in Mohajerin Esfahani et al. (2018), Heuberger (2004), Ahuja and Orlin (2001) and Burton and Toint (1992).

Let LP_c be a standard linear programming (LP) problem,

$$(LP_c) \quad \begin{aligned} \min \quad & cx \\ \text{s.t.} \quad & Ax = b, \\ & x \geq 0. \end{aligned} \tag{1.1}$$

where A is an $m \times n$ matrix and $m < n$, c^T and x are $n \times 1$ vectors and b is an $m \times 1$ vector.

As some combinatorial optimization problems can be described as LP problems, Zhang and Liu (1996) first studied the inverse LP problem under unit l_1 norm (ILP_{u1}). Let x^0 be a given feasible solution of the problem (LP). The aim of problem (ILP_{u1}) is to minimize the modification $\|\bar{c} - c\|$ under unit l_1 norm such that x^0 becomes an optimal solution of the modified problem ($LP_{\bar{c}}$). They transformed the problem (ILP_{u1}) into another LP problem and extended their results to the inverse bounded LP problem with a bounded variable constraint $l \leq x \leq u$. Furthermore, they applied their research methods to the inverse minimum cost flow problem and inverse assignment problem under unit l_1 norm. Zhang and Liu (1999) continued to consider a special case of problem (ILP_{u1}) in which the given feasible solution x^0 and one optimal solution of the original LP problem are 0-1 vectors. They gave a method based on dual theories for solving this special case and applied the method to the inverse shortest path problem and inverse assignment problem under unit l_1 norm. Huang and Liu (1999) also studied the inverse bounded LP problem and applied their results to the inverse minimum weight perfect k-matching problem on bipartite graphs under unit l_1 norm. Ahuja and Orlin (2001) studied the inverse canonical LP problem ($ICLP_1$) under weighted l_1 norm. They transformed the problem ($ICLP_1$) into an LP problem and transformed the inverse shortest path problem, inverse minimum cut problem, inverse minimum cost flow problem and inverse assignment problem under weighted l_1 norm into some minimum cost flow problems. Chan and Kaw (2020) concentrated on imputing unspecified constraint coefficient matrix A and a cost vector for a given linear optimization problem. Ghobadi and Mahmoudzadeh (2021) inferred the feasible region of LP problem that would render the given solutions feasible while making some optimal for the given cost function.

Ahmed and Guan (2005) studied the inverse optimal value problem ($IOVLP$) on LP. Given a desired optimal objective value K, and a set C of feasible cost vectors in an (LP), determine a cost vector $\bar{c} \in C$ such that the optimal objective value of the new problem $LP_{\bar{c}}$ is closest to the desired value K. They proved the problem ($IOVLP$) is NP-hard. Lv et al. (2008) and Lv et al. (2010) studied this problem under more general conditions using a nonlinear bilevel programming approach.

In this paper, we will study the restricted inverse optimal value problem ($RIOVLP_1$) on LP under weighted l_1 norm. Similar to the classical (ILP) problem, its objective is to minimize the modification $\|\bar{c}-c\|$ under weighted l_1 norm. Different to the classical (ILP) problem, in ($RIOVLP_1$) we not only require that the given feasible solution x^0 becomes an optimal solution of the problem ($LP_{\bar{c}}$) but also require that the optimal objective value $\bar{c}x^0$ equals the given value K. There are two main differences compared the problem ($RIOVLP_1$) with ($IOVLP$). One difference is on the optimization objectives. The problem ($RIOVLP_1$) aims to minimize the distance $\|\bar{c}-c\|$, while the problem ($IOVLP$) tries to minimize $|\bar{c}x^* - K|$ among $\bar{c} \in C$. The other difference is on the constraint conditions. In ($RIOVLP_1$) we impose a constraint on the optimal value $\bar{c}x^0$, which is equal to the given value K, while in ($IOVLP$), there is no cadidate solution for consideration.

Some restricted inverse optimal value problems on combinatorial optimization structures have been studied. Jia et al. (2023), Zhang et al. (2021), Wang et al. (2021) and Zhang et al. (2020) considered the restricted inverse optimal value problems on minimum spanning tree under different norms and proposed combinatorial algorithms in polynomial time. Zhang et al. (2023) studied the restricted inverse optimal value problem of shortest path on trees and devised an $O(n^2)$ algorithm under weighted l_1 norm and an $O(n)$ algorithm under unit l_1 norm. Zhang and Cai (1998) considered a more general restricted inverse optimal value problem under weighted l_1 norm on minimum cut which requires a set of cuts (not a cut) to become minimum cuts to make their objective value within a certain range (not equal to a given value). Cui and Hochbaum (2010) showed the restricted inverse optimal value problem on shortest path for general graphs is NP-hard when a collection of source-sink pairs with prescribed distances is given.

This paper is organized as follows. In section 2, we first formulate the problem ($RIOVLP_1$) as an LP problem by the dual theories. Then we analyze some properties of a sub-problem (D^z) of the dual ($RIOVLP_1$) problem with respect to a given value z. In section 3, we design a binary search algorithm to calculate the critical value z^* corresponding to an optimal solution of the problem ($RIOVLP_1$) by solving a sub-problem in each iteration. In section 4, we apply the above methods to the restricted inverse optimal value problems on Hitchcock and shortest path problem, respectively. Finally, we give some conclusions and further research in section 5.

2. Properties of the problem ($\boldsymbol{RIOVLP_1}$)

In this section, we study the restricted inverse optimal value problem on linear programming under weighted l_1 norm. We first formulated the problem as an LP problem, then analyze some properties of its sub-problems.

2.1. The mathematical model of the problem ($RIOVLP_1$)

Let x^0 be a given feasible solution, $\mathcal{F}_0 = \{x \in R^n | Ax = b, x \geq 0\}$ be the feasible region of the problem (LP) and K be a given real number. We aim to adjust the vector c to $\bar{c}$ such that x^0 becomes an optimal solution under $\bar{c}$ whose objective value $\bar{c}x^0$ equals K. Given a $1 \times n$ positive vector $d > 0$, the problem ($RIOVLP_1$) under weighted l_1 norm can be formulated as follows.

$$
\begin{aligned}
& & \min \quad & \sum_{j=1}^{n} d_j |\bar{c}_j - c_j| & \\
& (RIOVLP_1) & \text{s.t.} \quad & \min_{x \in \mathcal{F}_0} \bar{c}x = K, & (2.1) \\
& & & \bar{c}x^0 = K. & (2.2)
\end{aligned}
$$

It follows from the constraint (2.1) that the problem ($RIOVLP_1$) is not an LP problem. Fortunately, we can turn it into an LP problem by dual theories of LP. Next, we will explain the process in details.

Let us associate a dual variable $\pi \in R^m$ with the constraint (1.1). Then the dual problem of ($LP_{\bar{c}}$) can be stated as follows.

$$
\begin{aligned}
& & \max \quad & \pi b & \\
& (DLP_{\bar{c}}) & \text{s.t.} \quad & \pi A \leq \bar{c}. & (2.3)
\end{aligned}
$$

Let $J = \{j | x_j^0 = 0\}$, $\bar{J} = \{j | x_j^0 > 0\}$ and A_j be the j-th column of A.

Theorem 1. *If* $(\pi^*, \bar{c}^*)$ *is an optimal solution of the problem below,*

$$
\begin{aligned}
& & \min \quad & \sum_{j=1}^{n} d_j |\bar{c}_j - c_j| & \\
& (RIOVLP_1^1) & s.t. \quad & \pi A_j \leq \bar{c}_j, j \in J, & (2.4) \\
& & & \pi A_j = \bar{c}_j, j \in \bar{J}, & (2.5) \\
& & & \bar{c}x^0 = K. & (2.6)
\end{aligned}
$$

then $\bar{c}^*$ *is an optimal solution of the problem* ($RIOVLP_1$).

Proof. Let $\mathcal{F}$ and $\mathcal{F}_1$ be the feasible regions of the problems ($RIOVLP_1$) and ($RIOVLP_1^1$), respectively. Notice that the constraints (2.1)-(2.2) mean that x^0 is an optimal solution of the problem ($LP_{\bar{c}}$), which holds if and only if its dual problem ($DLP_{\bar{c}}$) has a feasible solution π which satisfies the **complementary slackness conditions** $x_j^0(\bar{c}_j - \pi A_j) = 0$ for any $j \in J \cup \bar{J}$. Suppose $\bar{c} \in \mathcal{F}$, then there exists π satisfying the constraints (2.4)-(2.5).

Hence, we have $(\pi, \bar{c}) \in \mathcal{F}_1$. On the other hand, suppose $(\pi^*, \bar{c}^*) \in \mathcal{F}_1$ is an optimal solution of the problem $(RIOVLP_1^1)$, then $(\pi^*, \bar{c}^*)$ satisfies the constraints (2.4)-(2.6), which renders that $\bar{c}^*$ satisfies the constraint (2.1)-(2.2). Hence, we have $\bar{c}^* \in \mathcal{F}$. Furthermore, $\bar{c}^*$ is an optimal solution of the problem $(RIOVLP_1)$, since the two problems have the same optimal objective value. □

Let $\bar{c}_j = c_j + \alpha_j - \beta_j$, where α_j, β_j are the increment and decrement of c_j, respectively. We claim that at least one of α_j and β_j is 0 based on the property of weighted l_1 norm for any $j \in J \cup \bar{J}$. Thus, the problem $(RIOVLP_1^1)$ can be turned into the model below.

$$
\begin{aligned}
& \min && \sum_{j \in J \cup \bar{J}} d_j(\alpha_j + \beta_j) & \\
& (RIOVLP_1^2) \text{ s.t.} && \pi A_j \leq c_j + \alpha_j - \beta_j, j \in J, & (2.7)\\
& && \pi A_j = c_j + \alpha_j - \beta_j, j \in \bar{J}, & (2.8)\\
& && \sum_{j \in \bar{J}} (c_j + \alpha_j - \beta_j) x_j^0 = K, & (2.9)\\
& && \alpha_j \geq 0, j \in J \cup \bar{J}, & (2.10)\\
& && \beta_j \geq 0, j \in J \cup \bar{J}. & (2.11)
\end{aligned}
$$

Associate a dual variable y_j with the constraints (2.7) and (2.8), and a dual variable z with the constraint (2.9). Then we can get its dual problem below.

$$
\begin{aligned}
& \max && \sum_{j \in J \cup \bar{J}} c_j y_j + \Big(K - \sum_{j \in \bar{J}} c_j x_j^0\Big) z & \\
& (DRIOVLP_1) \text{ s.t.} && Ay = 0, & (2.12)\\
& && -y_j \leq d_j, j \in J, & (2.13)\\
& && -y_j + x_j^0 z \leq d_j, j \in \bar{J}, & (2.14)\\
& && y_j \leq d_j, j \in J, & (2.15)\\
& && y_j - x_j^0 z \leq d_j, j \in \bar{J}, & (2.16)\\
& && y_j \leq 0, j \in J. & (2.17)
\end{aligned}
$$

Delete the item $(K - \sum_{j \in \bar{J}} c_j x_j^0) z$ from the objective function of the problem

$(DRIOVLP_1)$, we get a sub-problem (D^z) below.

$$
\begin{aligned}
\psi(z) := \quad & \max \quad \sum_{j\in J\cup\bar{J}} c_j y_j \\
(D^z) \quad & \text{s.t.} \quad Ay = 0, && (2.18)\\
& -y_j \le d_j, j \in J, && (2.19)\\
& -y_j + x_j^0 z \le d_j, j \in \bar{J}, && (2.20)\\
& y_j \le d_j, j \in J, && (2.21)\\
& y_j - x_j^0 z \le d_j, j \in \bar{J}, && (2.22)\\
& y_j \le 0, j \in J. && (2.23)
\end{aligned}
$$

Associate the dual variable $\bar{\pi}$ with the constraint (2.18), $\bar{\alpha}_j$ with the constraints (2.19 and 2.20), $\bar{\beta}_j$ with the constraints (2.21) and (2.22). If we treat z as a constant, then the dual problem of (D^z) can be stated as follows.

$$
\begin{aligned}
\phi(z) := \quad & \min \quad \sum_{j\in J\cup\bar{J}} d_j(\bar{\alpha}_j + \bar{\beta}_j) + z\sum_{j\in\bar{J}}(\bar{\beta}_j - \bar{\alpha}_j)x_j^0 \\
(P^z) \quad & \text{s.t.} \quad \bar{\pi}A_j - \bar{\alpha}_j + \bar{\beta}_j \le c_j, j \in J, && (2.24)\\
& \bar{\pi}A_j - \bar{\alpha}_j + \bar{\beta}_j = c_j, j \in \bar{J}, && (2.25)\\
& \bar{\alpha}_j \ge 0, j \in J \cup \bar{J}, && (2.26)\\
& \bar{\beta}_j \ge 0, j \in J \cup \bar{J}. && (2.27)
\end{aligned}
$$

Notice that the problem (P^z) can be regarded as a Lagrange relaxation of $(RIOVLP_1^2)$ by introducing a Lagrange multiplier z to the constraint (2.9).

Furthermore, let $y = x^0 - \breve{y}$, then the problem (D^z) can be turned into the following form.

$$
\begin{aligned}
& \min \quad \sum_{j\in J\cup\bar{J}} c_j \breve{y}_j - \sum_{j\in\bar{J}} c_j x_j^0 \\
(\breve{D}^z) \ & \text{s.t.} \quad A\breve{y} = b, && (2.28)\\
& 0 \le \breve{y}_j \le d_j, j \in J, && (2.29)\\
& (1-z)x_j^0 - d_j \le \breve{y}_j \le (1-z)x_j^0 + d_j, j \in \bar{J}. && (2.30)
\end{aligned}
$$

It is obvious that the problem $(\breve{D}^z)$ has the same objective function as (LP) in (1.1) by deleting the constant $-\sum_{j\in\bar{J}} c_j x_j^0$. As for the feasible region, they have the same constraint condition $A\breve{y} = b$. But they have different upper and lower bounds of the variables for a given z. Hence, it provides us a way to solve $(\breve{D}^z)$ similar to the original problem (LP).

2.2. Properties of the function $\psi(z)$

Let $\mathcal{F}_{D^z}$ be the feasible region of the problem (D^z) for a given $z \in \mathbb{R}$. Next, we will discuss some properties of the feasible region $\mathcal{F}_{D^z}$ and objective function $\psi(z)$ for a given $z \in \mathbb{R}$.

$(P1)$ For a given $z \in \mathbb{R}$, if $\mathcal{F}_{D^z} \neq \emptyset$, then the problem (D^z) has an optimal solution and $|\psi(z)| < +\infty$. It follows from $-d_j \leq y_j \leq 0$ for any $j \in J$ and $x_j^0 z - d_j \leq y_j \leq x_j^0 z + d_j$ for any $j \in \bar{J}$ by constraints (2.19)-(2.23).

$(P2)$ For any $z \in [\max_{j\in\bar{J}} \frac{-d_j}{x_j^0}, \min_{j\in\bar{J}} \frac{d_j}{x_j^0}]$, we have $\boldsymbol{y} = \boldsymbol{0} \in \mathcal{F}_{D^z} \neq \emptyset$ and $|\psi(z)| < +\infty$.

Theorem 2. *Suppose $z_1 < z_2$ and $|\psi(z_1)|, |\psi(z_2)| < +\infty$, then the function $\psi(z)$ has the following properties for any $z \in [z_1, z_2]$.*
(1) $|\psi(z)| < +\infty$.
(2) $\psi(z)$ is a continuous and piecewise function of z.
(3) $\psi(z)$ is a concave function of z.

Proof. We first claim that the assumption $|\psi(z_1)|, |\psi(z_2)| < +\infty$ for $z_1 < z_2$ must hold due to property $(P2)$. (1) Suppose $\tilde{\boldsymbol{y}}$ and $\bar{\boldsymbol{y}}$ are the optimal solutions of the problems (D^{z_1}) and (D^{z_2}) with objective values $\psi(z_1)$ and $\psi(z_2)$, respectively. Let $z = \lambda z_1 + (1-\lambda) z_2$, where $0 \leq \lambda \leq 1$. Next, we will prove $\lambda\tilde{\boldsymbol{y}} + (1-\lambda)\bar{\boldsymbol{y}}$ is a feasible solution of the problem (D^z). It follows from $A\tilde{y} = 0$ and $A\bar{y} = 0$ that $A(\lambda\tilde{\boldsymbol{y}} + (1-\lambda)\bar{\boldsymbol{y}}) = 0$ holds. It follows from $-\tilde{y}_j \leq d_j$ and $-\bar{y}_j \leq d_j$ that $-(\lambda\tilde{y}_j + (1-\lambda)\bar{y}_j) \leq d_j$ holds for any $j \in J$. Hence, the constraint (2.19) holds. In a similar way, the constraints (2.21) and (2.23) hold. It follows from $-\tilde{y}_j \leq d_j - x_j^0 z_1$ and $-\bar{y}_j \leq d_j - x_j^0 z_2$ that $-(\lambda\tilde{y}_j + (1-\lambda)\bar{y}_j) \leq \lambda(d_j - x_j^0 z_1) + (1-\lambda)(d_j - x_j^0 z_2) = d_j - x_j^0(\lambda z_1 + (1-\lambda) z_2) = d_j - x_j^0 z$ for any $j \in \bar{J}$. Hence, the constraint (2.20) holds. In a similar way, the constraint (2.22) holds. Therefore, we have $\mathcal{F}_{D^z} \neq \emptyset$ and $|\psi(z)| < +\infty$ for any $z \in [z_1, z_2]$ by property $(P1)$.

(2) We first claim that $\psi(z) = \phi(z)$ for any $z \in [z_1, z_2]$ based on the strong duality of linear programming. Furthermore, the problem (P^z) has an optimal solution $(\bar{\pi}^z, \bar{\alpha}^z, \bar{\beta}^z)$ since its dual problem (D^z) is feasible and bounded for any $z \in [z_1, z_2]$ by (1). Therefore, there exists an optimal solution $(\bar{\pi}^z, \bar{\alpha}^z, \bar{\beta}^z)$ satisfying $|\bar{\pi}_i^z|, |\bar{\alpha}_j^z|, |\bar{\beta}_j^z| \leq M$ for any $z \in [z_1, z_2]$, where $M = n! c_{max}$ with $c_{max} = \max_{j=1,\cdots,n} |c_j|$ is a large positive number by Lemma 2.1 in Papadimitriou and Steiglitz (1998). Hence, if we add a set of constraints: $|\bar{\pi}_i| \leq M$, $|\bar{\alpha}_j| \leq M$ and $|\bar{\beta}_j| \leq M$ to the problem (P^z), then the optimal solution of problem (P^z) will be unchanged. It renders that the feasible region of (P^z) becomes a bounded closed convex set. Hence, $\phi(z)$, as well as $\psi(z)$, is a continuous and piecewise function of z for $z \in [z_1, z_2]$.

(3) Suppose $z_l < z_r$ and $z_l, z_r \in [z_1, z_2]$. Let $(\bar{\pi}^*, \bar{\alpha}^*, \bar{\beta}^*)$ be an optimal solution of the problem (P^z), where $z = kz_l + (1-k)z_r$ and $0 \leq k \leq 1$. Then we have

$$\begin{aligned} & \phi(z) = \phi(kz_l + (1-k)z_r) \\ = \ & \sum_{j \in J \cup \bar{J}} d_j(\bar{\alpha}_j^* + \bar{\beta}_j^*) + (kz_l + (1-k)z_r) \sum_{j \in \bar{J}} (\bar{\beta}_j^* - \bar{\alpha}_j^*) x_j^0 \\ = \ & k \left(\sum_{j \in J \cup \bar{J}} d_j(\bar{\alpha}_j^* + \bar{\beta}_j^*) + z_l \sum_{j \in \bar{J}} (\bar{\beta}_j^* - \bar{\alpha}_j^*) x_j^0 \right) \\ + \ & (1-k) \left(\sum_{j \in J \cup \bar{J}} d_j(\bar{\alpha}_j^* + \bar{\beta}_j^*) + z_r \sum_{j \in \bar{J}} (\bar{\beta}_j^* - \bar{\alpha}_j^*) x_j^0 \right) \\ \geq \ & k\phi(z_l) + (1-k)\phi(z_r). \end{aligned}$$

Hence, $\phi(z)$, as well as $\psi(z)$, is a concave function of z. □

Now we claim that the feasible region $\mathcal{F}_{D^z}$ may be empty for a given $z \in \mathbb{R}$. Next we present an example to illustrate it. Let $A = [E_m, E_m]$ and $z_0 = \frac{2d_{max}}{x^0_{min}}$, where E_m is an identity matrix, $x^0_{min} = \min_{j \in \bar{J}} x_j^0$ and $d_{max} = \max_{j \in J \cup \bar{J}} d_j$. Then $x_j^0 z - d_j \geq x_j^0 z - d_{max} > x_j^0 \frac{2d_{max}}{x^0_{min}} - d_{max} \geq 2d_{max} - d_{max} = d_{max}$ for any $j \in \bar{J}$ when $z > z_0$. Therefore, we have $d_{max} < x_j^0 z - d_j \leq y_j \leq x_j^0 z + d_j$ for any $j \in \bar{J}$. Consider the i-th constraint in (2.18), we have $y_i + y_{i+m} = 0$. If $i \in \bar{J}$ or $i + m \in \bar{J}$, then this constraint can not hold since $-d_j \leq y_j \leq 0$ for any $j \in J$ and $y_j > d_{max}$ for any $j \in \bar{J}$. Unfortunately, this case must exist since $\bar{J} \neq \emptyset$. Hence, $\mathcal{F}_{D^z}$ may be an empty set for a given $z > 0$. In a similar way, $\mathcal{F}_{D^z}$ may be an empty set for a given $z < 0$. In these cases, $\psi(z)$ is undefined.

Theorem 3. *(1) If $z_1 < 0$ and $\mathcal{F}_{D^{z_1}} = \emptyset$, then $\mathcal{F}_{D^z} = \emptyset$ for any $z \le z_1$.*
(2) If $z_2 > 0$ and $\mathcal{F}_{D^{z_2}} = \emptyset$, then $\mathcal{F}_{D^z} = \emptyset$ for any $z \ge z_2$.

PROOF. (1) Suppose there exists $z_0 < z_1$ satisfying $\mathcal{F}_{D^{z_0}} \ne \emptyset$. Then we have $\mathcal{F}_{D^z} \ne \emptyset$ for any $z \in [z_0, 0]$ by Theorem 2, which contradicts that $\mathcal{F}_{D^{z_1}} = \emptyset$. Hence, the conclusion holds. In a similar way, (2) holds. □

For simplicity of discussion, we turn the problem (D^z) into a standard LP problem by the following steps. (i) It follows from the constraint (2.23) and $d_j > 0$ that (2.21) can be omitted. (ii) Add relaxation variables ξ_j to the constraints (2.19) and (2.20). (iii) Add relaxation variables η_j to the constraints (2.22) and (2.23). (iv) Replace non-constrained variables y_j by two non-negative variables $\dot{y}_j$ and $\ddot{y}_j$. Then we get its maximization standard LP problem below.

$$\begin{aligned}
\psi(z) := \quad & \max \quad \sum_{j\in J\cup\bar{J}} c_j(\dot{y}_j - \ddot{y}_j) & \\
(D_s^z) \quad & \text{s.t.} \quad A\dot{y} - A\ddot{y} = 0, & (2.31)\\
& -\dot{y}_j + \ddot{y}_j + \xi_j = d_j, j \in J, & (2.32)\\
& -\dot{y}_j + \ddot{y}_j + \xi_j = d_j - x_j^0 z, j \in \bar{J}, & (2.33)\\
& \dot{y}_j - \ddot{y}_j + \eta_j = 0, j \in J, & (2.34)\\
& \dot{y}_j - \ddot{y}_j + \eta_j = d_j + x_j^0 z, j \in \bar{J}, & (2.35)\\
& \dot{y}, \ddot{y}, \xi, \eta \ge 0. & (2.36)
\end{aligned}$$

Let $\bar{A}$, $\bar{b}^z$, $\bar{c}$ and $\bar{X}$ be the coefficient matrix, right-hand vector, cost coefficient vector and all variables of the problem (D_s^z), respectively. For the convenience of further discussion, we assume $J = \{1, 2, \cdots, k\}$ and $\bar{J} = \{k+1, \cdots, n\}$, where $0 \le k \le n-1$. Then we have

$$\bar{A} = \begin{bmatrix} A & -A & \mathbf{0} & \mathbf{0} \\ -E_n & E_n & E_n & \mathbf{0} \\ E_n & -E_n & \mathbf{0} & E_n \end{bmatrix}_{\bar{m}\times\bar{n}}, \tag{2.37}$$

$\bar{c} = (c, -c, \mathbf{0}, \mathbf{0})$, $\bar{b}^z = (\mathbf{0}, d_J, d_{\bar{J}} - zx^0_{\bar{J}}, \mathbf{0}, d_{\bar{J}} + zx^0_{\bar{J}})^T$, $\bar{X}^T = (\dot{y}, \ddot{y}, \xi, \eta)$, where $\bar{m} = 2n + m, \bar{n} = 4n$ and $x^0_{\bar{J}}$ is a row vector here. The corresponding standard LP model can be formulated below.

$$\begin{aligned} \psi(z) := \quad & \max \quad \bar{c}\bar{X} \\ (D^z_s) \quad & \text{s.t.} \quad \bar{A}\bar{X} = \bar{b}^z, \\ & \qquad \bar{X} \geq 0. \end{aligned}$$

Next we need to clarify the maximum value z_l and minimum value z_r satisfying the property (1) and (2) in Theorem 3, respectively. For convenience, we call them left and right break points, respectively.

Definition 1. *A value $z_l < 0$ is called* ***the left break point*** *of $\psi(z)$, if $\mathcal{F}_{D^z} = \emptyset$ for any $z < z_l$ and $\mathcal{F}_{D^z} \neq \emptyset$ for any $z \in [z_l, 0]$. A value $z_r > 0$ is called* ***the right break point*** *of $\psi(z)$, if $\mathcal{F}_{D^z} = \emptyset$ for any $z > z_r$ and $\mathcal{F}_{D^z} \neq \emptyset$ for any $z \in [0, z_r]$.*

Theorem 4. *(1) If there is a value $z_1 < 0$ satisfying $\mathcal{F}_{D^{z_1}} = \emptyset$, then there exists a left break point z_l. (2) If there is a value $z_2 > 0$ satisfying $\mathcal{F}_{D^{z_2}} = \emptyset$, then there exists a right break point z_r. (3) For any $z \geq z_l$ and $z \leq z_r$, $|\psi(z)| < +\infty$.*

PROOF. (1) Note that $\mathcal{F}_{D^z} \neq \emptyset$ for any $z \in [\max_{j\in\bar{J}} \frac{-d_j}{x^0_j}, \min_{j\in\bar{J}} \frac{d_j}{x^0_j}]$ by property (P2). Hence, we can assume that there exists $z_0 < 0$ satisfying $\mathcal{F}_{D^{z_0}} \neq \emptyset$. Let B be the optimal basis corresponding to the problem $(D^{z_0}_s)$. Consider the i-th basic variable: $\bar{X}^{z_0}_i = B^{-1}_i \bar{b}^{z_0} = k_i z_0 + g_i$, where B^{-1}_i is the i-th row of the inverse matrix B^{-1} and k_i, g_i are constants. Therefore, when the optimal basis B changes to a non-optimal basis as z_0 decreases, there must be a non-negative basic variable that changes to a negative value. Suppose z_l is such a value. Then we have $\mathcal{F}_{D^{z_l}} \neq \emptyset$. Furthermore, it follows from property (1) in Theorem 2 that $\mathcal{F}_{D^z} \neq \emptyset$ for any $z \in [z_l, 0]$. Hence, z_l is the left break point. The property (2) holds similarly. (3) This property holds due to (1) and (2). □

Based on the previous discussions, we claim that there are four cases of the left and right break points in the function $\psi(z)$ as shown in Figure 1, where z_l and z_r represent the left and right break points, respectively.

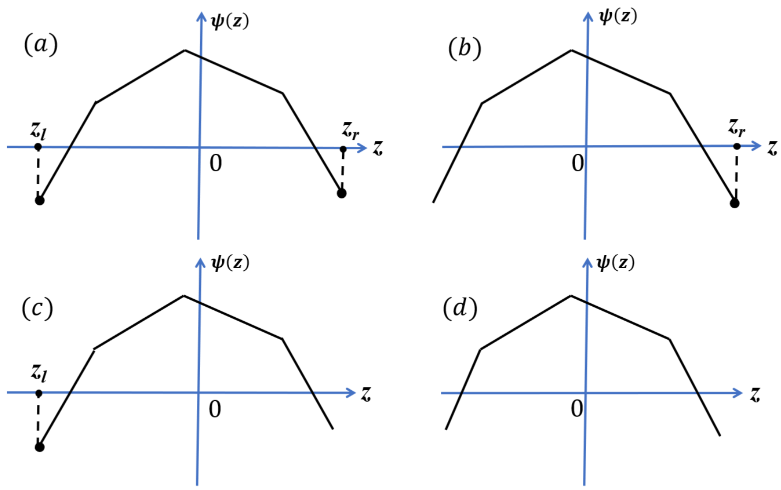

Figure 1: Four cases of left and right break points in the function $\psi(z)$.
(a) z_l, z_r exist; (b) z_r exists; (c) z_l exists; (d) no z_l, z_r.

Notice that $\psi(z)$ is a continuous piecewise linear function, we introduce the following definition.

Definition 2. *Let $(z_i, \psi(z_i))$ be an intersection point of two adjacent lines. We call z_i* ***a turning coordinate****.*

3. Solve the problem $(RIOVLP_1)$ when A is unimodular

In this section, we solve the problem $(RIOVLP_1)$ when A is a unimodular coefficient matrix. Firstly we analyze some properties of turning coordinates. Secondly, we present an important theorem to determine the critical value z^* in an optimal solution (y^*, z^*) of the problem $(DRIOVLP_1)$. Thirdly, we calculate the slope k_z of a piece of segment in $\psi(z)$ for a given z which is not a turning coordinate. Then we present two algorithms to calculate the left and right break points z_l and z_r. Finally, we propose an algorithm to solve the problem $(RIOVLP_1)$.

Assumption 1. *The coefficient matrix A is unimodular.*

Assumption 2. *The given feasible solution x^0 and the weight vector d are integral.*

3.1. Properties of turning coordinates

To describe the form of left and right break points, as well as turning points, we present the following lemma.

Lemma 1. *Suppose Assumptions 1 and 2 hold. If z_i is a turning coordinate, then z_i is in the form of $\frac{\varsigma}{\sigma}$, where ς, σ are integers and $|\sigma| \leq 2|\bar{J}|x^0_{max}$, $|\varsigma| \leq (n + |\bar{J}|)d_{max}$, $x^0_{max} = \max_{j\in\bar{J}} x^0_j$.*

Proof. It follows from Assumption 1 that the coefficient matrix $\bar{A}$ defined as in (2.37) of the problem (D^z_s) is also a unimodular matrix. For a given $z \in \mathbb{R}$, if $\mathcal{F}_{D^z} \neq \emptyset$, then there exists an optimal basis matrix B of the problem (D^z_s). Hence, the elements $b^-_{ij} \in \{0, 1, -1\}$ for the inverse matrix $B^{-1} = (b^-_{ij})$. Specifically, let $b^1_{ij}, b^2_{ij}, b^3_{ij}$ be the elements of B^{-1}_i corresponding to j in (2.32), (2.33) and (2.35), respectively. Let $\bar{X}^z_i$ be the i-th basic variable corresponding to B. Then we have

$$\begin{aligned}\bar{X}^z_i &= B^{-1}_i \bar{b}^z \\ &= \sum_{j\in J} b^1_{ij} d_j + \sum_{j\in\bar{J}} b^2_{ij}(d_j - x^0_j z) + \sum_{j\in\bar{J}} b^3_{ij}(d_j + x^0_j z) \\ &= \sum_{j\in J} b^1_{ij} d_j + \sum_{j\in\bar{J}} (b^2_{ij} d_j + b^3_{ij} d_j) + z \sum_{j\in\bar{J}} x^0_j (b^3_{ij} - b^2_{ij}).\end{aligned}$$

With the variation of z, the optimal basis matrix B of the problem (D^z_s) will change at a value in the form of $z = \frac{\sum_{j\in J} b^1_{ij} d_j + \sum_{j\in\bar{J}}(b^2_{ij} d_j + b^3_{ij} d_j)}{\sum_{j\in\bar{J}} x^0_j (b^2_{ij} - b^3_{ij})}$ as in the proof of Theorem 4. It is easy to know $|\sum_{j\in J} b^1_{ij} d_j + \sum_{j\in\bar{J}}(b^2_{ij} d_j + b^3_{ij} d_j)| \leq (n + |\bar{J}|)d_{max}$ is an integer. Furthermore, $|\sum_{j\in\bar{J}} x^0_j (b^2_{ij} - b^3_{ij})| \leq 2|\bar{J}|x^0_{max}$ is also an integer. As a conclusion, the lemma holds. □

Corollary 1. *Suppose z_i, z_j are two turning coordinates, then $|z_i - z_j| \geq \frac{1}{(2|\bar{J}|x^0_{max})^2}$.*

Proof. Suppose $z_i = \frac{\varsigma_i}{\sigma_i}$ and $z_j = \frac{\varsigma_j}{\sigma_j}$, then $\varsigma_i, \varsigma_j, \sigma_i, \sigma_j \in \mathbb{Z}$ and $|\sigma_i|, |\sigma_j| \leq 2|\bar{J}|x^0_{max}$. Hence,

$$\left|\frac{\varsigma_i}{\sigma_i} - \frac{\varsigma_j}{\sigma_j}\right| = \left|\frac{\varsigma_i\sigma_j - \varsigma_j\sigma_i}{\sigma_i\sigma_j}\right| \geq \frac{|\varsigma_i\sigma_j - \varsigma_j\sigma_i|}{(2|\bar{J}|x^0_{max})^2} \geq \frac{1}{(2|\bar{J}|x^0_{max})^2}.$$

This completes the proof. □

Let $TC = \{\frac{\varsigma}{\sigma}|\varsigma, \sigma \in \mathbb{Z}, |\sigma| \leq 2|\bar{J}|x^0_{max}, |\varsigma| \leq (n+|\bar{J}|)d_{max}\}$, then a turning coordinate must in TC, but the opposite is not true.

It follows from Lemma 1 that the optimal basis of the problem $(D^{z_0}_s)$ keeps unchanged in its ϵ-neighborhood for a given value $z_0 \notin TC$ and $\epsilon > 0$. It renders that the optimal solution $(\bar{\pi}^*, \bar{\alpha}^*, \bar{\beta}^*)$ of the problem (P^{z_0}) keeps unchanged for $z \in [z_0 - \epsilon, z_0 + \epsilon]$. Therefore, the slope of $\psi(z)$ is $k_z = \sum_{j \in \bar{J}}(\bar{\beta}^*_j - \bar{\alpha}^*_j)x^0_j$ for $z \in [z_0 - \epsilon, z_0 + \epsilon]$.

Corollary 2. *For a given value $z_0 \notin TC$, if $\mathcal{F}_{D^{z_0}} \neq \emptyset$ and B is an optimal basis of the problem $(D^{z_0}_s)$, then there must exist $\epsilon > 0$ satisfying that B is also an optimal basis of the problem (D^z_s) for any $z \in [z_0 - \epsilon, z_0 + \epsilon]$.*

3.2. Solve the dual inverse problem (DRIOVLP$_1$)

For the convenience of following discussion, we give the definition below.

Definition 3. *Suppose (y^*, z^*) is an optimal solution of the problem (DRIO VLP$_1$), then we call z^** ***a critical value****.*

If we can determine **a critical value** z^*, then we can solve the sub-problem (D^{z^*}) and obtain its optimal solution y^*, which renders an optimal solution (y^*, z^*) of the problem (*DRIOVLP*$_1$). Hence, the main issue now is how to determine z^*. To do so, we need to determine two critical consecutive segments in the piecewise linear function $\psi(z)$, whose slopes k_1, k_2 satisfy $k_2 < c_{\bar{J}}x^0_{\bar{J}} - K \leq k_1$. Notice that there is only one slope of a linear function for the left and right break points and there are cases that the left and right break points do not exist as shown in Figure 1. We first introduce the minimum and maximum turning coordinates $\underline{z}$ and $\bar{z}$ to substitute the the left and right break points. Then define the right slope of $\psi(z)$ (if exists) as k^+_z and the left slope of $\psi(z)$ (if exists) as k^-_z.

Next, we propose an optimality condition to determine z^* at each of the four cases. Figure 2 shows the values $\underline{z}$, $\bar{z}$ and z^* in each case, where the red line represents the line with the slope $\delta = c_{\bar{J}}x^0_{\bar{J}} - K$ in each subcases.

Theorem 5. *Suppose $(\hat{z}, \psi(\hat{z}))$ is an intersection point of two consecutive segments in the piecewise linear function $\psi(z)$.*

(1) *Suppose $\mathcal{F}_{D^z} = \emptyset$ for any $z \in (-\infty, \underline{z}) \cup (\bar{z}, +\infty)$. Then*

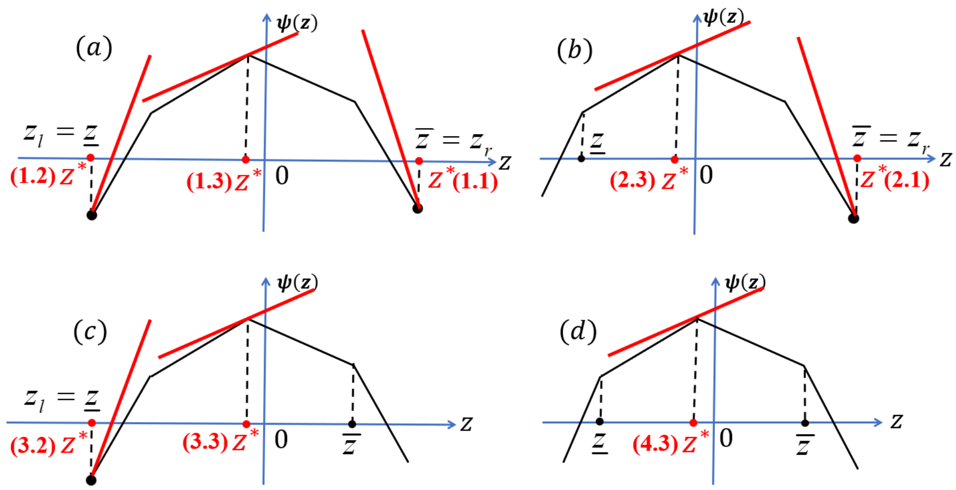

Figure 2: The values $\underline{z}$, $\bar{z}$ and z^* in each case of Theorem 5.

(1.1) *If* $k^-_{\bar{z}} \geq \delta$, *then* $z^* = \bar{z}$ *is* ***the critical value****.*
(1.2) *If* $k^+_{\underline{z}} < \delta$, *then* $z^* = \underline{z}$ *is* ***the critical value****.*
(1.3) *If* $k^+_{\hat{z}} < \delta \leq k^-_{\hat{z}}$, *then* $z^* = \hat{z}$ *is* ***the critical value****.*
(2) *Suppose* $\mathcal{F}_{D^z} \neq \emptyset$ *for any* $z \in (-\infty, \underline{z})$ *and* $\mathcal{F}_{D^z} = \emptyset$ *for any* $z \in (\bar{z}, +\infty)$. *Then*
(2.1) *If* $k^-_{\bar{z}} \geq \delta$, *then* $z^* = \bar{z}$ *is* ***the critical value****.*
(2.2) *If* $k^-_{\underline{z}} < \delta$, *then the problem (RIOVLP*$_1$*) is infeasible.*
(2.3) *If* $k^+_{\hat{z}} < \delta \leq k^-_{\hat{z}}$, *then* $z^* = \hat{z}$ *is* ***the critical value****.*
(3) *Suppose* $\mathcal{F}_{D^z} = \emptyset$ *for any* $z \in (-\infty, \underline{z})$ *and* $\mathcal{F}_{D^z} \neq \emptyset$ *for any* $z \in (\bar{z}, +\infty)$. *Then*
(3.1) *If* $k^+_{\bar{z}} > \delta$, *then the problem (RIOVLP*$_1$*) is infeasible.*
(3.2) *If* $k^+_{\underline{z}} \leq \delta$, *then* $z^* = \underline{z}$ *is* ***the critical value****.*
(3.3) *If* $k^+_{\hat{z}} \leq \delta < k^-_{\hat{z}}$, *then* $z^* = \hat{z}$ *is* ***the critical value****.*
(4) *Suppose* $\mathcal{F}_{D^z} \neq \emptyset$ *for any* $z \in (-\infty, \underline{z}) \cup (\bar{z}, +\infty)$. *Then*
(4.1) *If* $k^+_{\bar{z}} > \delta$, *then the problem (RIOVLP*$_1$*) is infeasible.*
(4.2) *If* $k^-_{\underline{z}} < \delta$, *then the problem (RIOVLP*$_1$*) is infeasible.*
(4.3) *If* $k^+_{\hat{z}} \leq \delta \leq k^-_{\hat{z}}$, *then* $z^* = \hat{z}$ *is* ***the critical value****.*

Proof. (1) (1.1) Suppose $(\tilde{y}, \tilde{z})$ and $\bar{y}$ are optimal solutions of the problems $(DRIOVLP_1)$ and $(D^{\bar{z}})$, respectively. It is easy to know that $\tilde{z} \leq \bar{z}$. If

$\tilde{z} < \bar{z}$, then

$$
\begin{aligned}
& \sum_{j \in J \cup \bar{J}} c_j \tilde{y}_j + (K - \sum_{j \in \bar{J}} c_j x_j^0)\tilde{z} = c\tilde{y} - \delta \tilde{z} \\
\leq \ & \psi(\tilde{z}) - \delta \tilde{z} \\
\leq \ & k_{\bar{z}}^{-}(\tilde{z} - \bar{z}) + \psi(\bar{z}) - \delta \tilde{z}, \ (\psi(z) \ is \ concave \ by \ Theorem \ 2.) \\
= \ & \psi(\bar{z}) - \delta \bar{z} + (\tilde{z} - \bar{z})(-\delta + k_{\bar{z}}^{-}) \\
\leq \ & \psi(\bar{z}) - \delta \bar{z} \\
= \ & c\bar{y} - \delta \bar{z}.
\end{aligned}
$$

Hence, $(\bar{y}, \bar{z})$ is also an optimal solution of the problem $(DRIOVLP_1)$ if the equalities always hold in the above three inequalities. Otherwise, it contradicts the optimality of $(\tilde{y}, \tilde{z})$.

(1.2) Suppose $(\tilde{y}, \tilde{z})$ and $\underline{y}$ are optimal solutions of $(DRIOVLP_1)$ and $(D^{\underline{z}})$, respectively. It is easy to know that $\tilde{z} \geq \underline{z}$. If $\tilde{z} > \underline{z}$, then

$$
\begin{aligned}
& c\tilde{y} - \delta \tilde{z} \\
\leq \ & \psi(\tilde{z}) - \delta \tilde{z} \\
\leq \ & k_{\underline{z}}^{+}(\tilde{z} - \underline{z}) + \psi(\underline{z}) - \delta \tilde{z}, \ (\psi(z) \ is \ concave \ by \ Theorem \ 2.) \\
= \ & \psi(\underline{z}) - \delta \underline{z} + (\tilde{z} - \underline{z})(-\delta + k_{\underline{z}}^{+}) \\
< \ & \psi(\underline{z}) - \delta \underline{z} \\
= \ & c\underline{y} - \delta \underline{z}.
\end{aligned}
$$

which contradicts that $(\tilde{y}, \tilde{z})$ is an optimal solution of $(DRIOVLP_1)$.

(1.3) Suppose $(\tilde{y}, \tilde{z})$ and $\hat{y}$ are optimal solutions of $(DRIOVLP_1)$ and

$(D^{\hat{z}})$, respectively. We will discuss it in two situations. (i) If $\tilde{z} > \hat{z}$, then

$$\begin{aligned} & c\tilde{y} - \delta\tilde{z} \\ \leq\ & \psi(\tilde{z}) - \delta\tilde{z} \\ \leq\ & k^{+}_{\hat{z}}(\tilde{z} - \hat{z}) + \psi(\hat{z}) - \delta\tilde{z},\ (\psi(z) \textit{ is concave by Theorem } 2.) \\ =\ & \psi(\hat{z}) - \delta\hat{z} + (\tilde{z} - \hat{z})(-\delta + k^{+}_{\hat{z}}) \\ <\ & \psi(\hat{z}) - \delta\hat{z} \\ =\ & c\hat{y} - \delta\hat{z}. \end{aligned}$$

which contradicts that $(\tilde{y}, \tilde{z})$ is an optimal solution of the problem (DRI-$OVLP_1$).

(ii) If $\tilde{z} < \hat{z}$, then

$$\begin{aligned} & c\tilde{y} - \delta\tilde{z} \\ \leq\ & \psi(\tilde{z}) - \delta\tilde{z} \\ \leq\ & k^{-}_{\hat{z}}(\tilde{z} - \hat{z}) + \psi(\hat{z}) - \delta\tilde{z},\ (\psi(z) \textit{ is concave by Theorem } 2.) \\ =\ & \psi(\hat{z}) - \delta\hat{z} + (\tilde{z} - \hat{z})(-\delta + k^{-}_{\hat{z}}) \\ \leq\ & \psi(\hat{z}) - \delta\hat{z} \\ =\ & c\hat{y} - \delta\hat{z}. \end{aligned}$$

Then $(\hat{y}, \hat{z})$ is also an optimal solution of problem ($DRIOVLP_1$) if the equalities always hold in the above three inequalities. Otherwise, it contradicts the optimality of $(\tilde{y}, \tilde{z})$.

Notice that one and only one case holds due to the concavity of $\psi(z)$.

In a similar way, we can show that (2),(3) and (4) hold. □

3.3. Calculate the slope k_z for a given $z \notin TC$

In this subsection, we focus on addressing the following two questions. For a given $z \in \mathbb{R}$, (Q1) if $\mathcal{F}_{D^z} \neq \emptyset$, then how to determine whether z is a turning coordinate of an intersection point $(z, \psi(z))$ of two consecutive segments in $\psi(z)$? (Q2) if $z \notin TC$ and $\mathcal{F}_{D^z} \neq \emptyset$, then how to calculate the slope k_z and obtain the left and right turning coordinates closest to z? To solve these problems, we make the assumption below.

Assumption 3. *There is an algorithm $\mathcal{A}_z$ not only to determine whether $\mathcal{F}_{D^z}$ is an empty set for a given $z \in \mathbb{R}$ but also to solve the problem (D^z) or (D_s^z) or $(\breve{D}^z)$ and calculate $\psi(z)$ when $\mathcal{F}_{D^z} \neq \emptyset$.*

Let $\Delta = \frac{1}{(2|\bar{J}|x^0_{max})^2}$. To answer the question (Q1), the following lemma is clearly valid.

Lemma 2. *Suppose Assumption 3 holds and $\mathcal{F}_{D^z} \neq \emptyset$. Let $z_1 = z - \frac{\Delta}{2}$, $z_2 = z - \frac{\Delta}{4}$, $z_3 = z + \frac{\Delta}{4}$, $z_4 = z + \frac{\Delta}{2}$, $k_{z,z_1} = \frac{\psi(z)-\psi(z_1)}{z-z_1}$, $k_{z,z_2} = \frac{\psi(z)-\psi(z_2)}{z-z_2}$, $k_{z,z_3} = \frac{\psi(z)-\psi(z_3)}{z-z_3}$ and $k_{z,z_4} = \frac{\psi(z)-\psi(z_4)}{z-z_4}$. If $k_{z,z_1} = k_{z,z_2}$, $k_{z,z_3} = k_{z,z_4}$ and $k_{z,z_2} \neq k_{z,z_3}$, then z is a turning coordinate of an intersection point $(z, \psi(z))$ of two consecutive segments in the function $\psi(z)$.*

Next, we concentrate on (Q2) calculating the slope k_z for a given $z \notin TC$ and $\mathcal{F}_{D^z} \neq \emptyset$. It seems straightforward to calculate $k_z = \frac{\psi(z+\epsilon)-\psi(z)}{\epsilon}$, where ϵ is a very small positive real number. However, it is a dilemma to determine ϵ such that $(z, z+\epsilon)$ does not contain a turning coordinate. Hence, we calculate k_z by duality theory of *LP*. We can obtain an optimal solution $(\bar{\pi}^*, \bar{\alpha}^*, \bar{\beta}^*)$ of the problem (P^z) based on an optimal solution y^* of the problem (D^z). Therefore, we have $k_z = \sum_{j\in\bar{J}}(\bar{\beta}^*_j - \bar{\alpha}^*_j)x^0_j$ by Corollary 2.

If $\mathcal{F}_{D^{z_0}} \neq \emptyset$ and $z_0 \notin TC$, then we will discuss how to calculate the left and right turning coordinates closest to z_0 in detail. First, we choose $z_1 \in \mathbb{R}$ satisfying the conditions (C1), (C2) or (C1), (C3) below.

(C1) $z_1 \neq z_0$ and $z_1 \notin TC$. (C2) If $z_1 < z_0$, then $[z_1, z_0] \cap TC = \emptyset$. (C3) If $z_1 > z_0$, then $[z_0, z_1] \cap TC = \emptyset$.

Suppose y^0, y^1 are the optimal solutions of problems (D^{z_0}) and (D^{z_1}), respectively and z_0^l, z_0^r are the left and right turning coordinates closest to z_0, respectively. Let $k_j = \frac{y_j^0 - y_j^1}{z_0 - z_1}$ and $y_j^z = k_j(z - z_0) + y_j^0$, $j \in J \cup \bar{J}$. Notice that (D_s^z) has the same optimal basis for any $z \in [z_0^l, z_0^r]$. Therefore, y^z is the optimal solution of the problem (D^z) for any $z \in [z_0^l, z_0^r]$ if and only if y^z satisfies the constraints (2.19)-(2.23), which can be formulated as $-d_j \leq y_j^z \leq 0$ for $j \in J$ and $x_j^0 z - d_j \leq y_j^z \leq x_j^0 z + d_j$ for $j \in \bar{J}$. Therefore, we have the following properties. 1) if $j \in J$ and $k_j > 0$, then $\frac{k_j z_0 - d_j - y_j^0}{k_j} \leq$

$z \le \frac{k_j z_0 - y_j^0}{k_j}$. 2) if $j \in J$ and $k_j < 0$, then $\frac{k_j z_0 - y_j^0}{k_j} \le z \le \frac{k_j z_0 - d_j - y_j^0}{k_j}$. 3) if $j \in \bar{J}$ and $k_j > x_j^0$, then $\frac{k_j z_0 - d_j - y_j^0}{k_j - x_j^0} \le z \le \frac{k_j z_0 + d_j - y_j^0}{k_j - x_j^0}$. 4) if $j \in \bar{J}$ and $k_j < x_j^0$, then $\frac{k_j z_0 + d_j - y_j^0}{k_j - x_j^0} \le z \le \frac{k_j z_0 - d_j - y_j^0}{k_j - x_j^0}$. Let $z_0^1 = \max_{j \in J, k_j > 0} \frac{k_j z_0 - d_j - y_j^0}{k_j}$, $z_0^2 = \max_{j \in J, k_j < 0} \frac{k_j z_0 - y_j^0}{k_j}$, $z_0^3 = \max_{j \in \bar{J}, k_j > x_j^0} \frac{k_j z_0 - d_j - y_j^0}{k_j - x_j^0}$, $z_0^4 = \max_{j \in \bar{J}, k_j < x_j^0} \frac{k_j z_0 + d_j - y_j^0}{k_j - x_j^0}$, $z_0^5 = \min_{j \in J, k_j > 0} \frac{k_j z_0 - y_j^0}{k_j}$, $z_0^6 = \min_{j \in J, k_j < 0} \frac{k_j z_0 - d_j - y_j^0}{k_j}$, $z_0^7 = \min_{j \in \bar{J}, k_j > x_j^0} \frac{k_j z_0 + d_j - y_j^0}{k_j - x_j^0}$ and $z_0^8 = \min_{j \in \bar{J}, k_j < x_j^0} \frac{k_j z_0 - d_j - y_j^0}{k_j - x_j^0}$. Hence, we have

$$z_0^l = \max\{z_0^1, z_0^2, z_0^3, z_0^4\}, \quad z_0^r = \min\{z_0^5, z_0^6, z_0^7, z_0^8\}. \tag{3.38}$$

Based on the above analysis, we have the following lemma.

Lemma 3. *Let $k_j = \frac{y_j^0 - y_j^1}{z_0 - z_1}$ and $y_j^z = k_j(z - z_0) + y_j^0$, $j \in J \cup \bar{J}$. Then y^z is the optimal solution of the problem (D^z) for any $z \in [z_0^l, z_0^r]$, where z_0^l, z_0^r are defined as in (3.38).*

3.4. Calculate the left and right break points z_l and z_r

In this subsection, we present two algorithms **LBP** and **RBP** to calculate the left and right break points z_l and z_r (if exist), respectively. As they are similar, we only discuss the main idea to calculate z_l in detail.

Firstly, we check the existence of the left break point z_l. Initialize $\underline{z}_l := -(n+|\bar{J}|)d_{max} - 1$ and $\bar{z}_l := 0$. If $\mathcal{F}_{D^{\underline{z}_l}} \neq \emptyset$, then the left break point z_l dose not exist. Otherwise, we can use a binary search method to determine an interval $[\underline{z}_l, \bar{z}_l]$ which satisfies $\bar{z}_l - \underline{z}_l < \frac{\Delta}{2}$ and $\mathcal{F}_{D^{\underline{z}_l}} = \emptyset$, $\mathcal{F}_{D^{\bar{z}_l}} \neq \emptyset$. Secondly, let $z_0 = \bar{z}_l$ and $z_1 = z_0 + \frac{\Delta}{2}$, then we have $\mathcal{F}_{D^{z_1}} \neq \emptyset$ and z_0, z_1 satisfy the conditions (C1) and (C3) in subsection 3.3. Finally, calculate the optimal solutions y^0, y^1 of the problems (D^{z_0}) and (D^{z_1}). Then we can get $z_l = z_0^l$ by (3.38).

Algorithm 1 z_l=**LBP**(A, b, c, d, x^0, K).

Input: The coefficient matrix A, the vectors b, c, d, x^0 and a value K.

Output: The left break point z_l.

Let $\underline{z}_l := -(n+|\bar{J}|)d_{max} - 1$, $\bar{z}_l := 0$ and $\Delta := \frac{1}{(2|\bar{J}|x^0_{max})^2}$.
if $\mathcal{F}_{D^{\bar{z}_l}} \neq \emptyset$ **then**
  return $z_l := -\infty$.
else
  while $\bar{z}_l - \underline{z}_l \geq \frac{\Delta}{2}$ **do**
    Let $z := \frac{\underline{z}_l + \bar{z}_l}{2}$.
    if $\mathcal{F}_{D^z} \neq \emptyset$ **then**
      $\bar{z}_l := z$.
    else
      $\underline{z}_l := z$.
    end if
  end while
  Let $z_1 := \bar{z}_l + \frac{\Delta}{2}$, $z_0 := \bar{z}_l$ and y^0, y^1 be the optimal solutions of the problems (D^{z_0}) and (D^{z_1}), respectively.
  return $z_l := z_0^l$ by (3.38).
end if

Next, we give the algorithm below to calculate z_r similar to algorithm 1.

Algorithm 2 z_r=**RBP**(A, b, c, d, x^0, K).

Input: The coefficient matrix A, the vectors b, c, d, x^0 and a value K.
Output: The right break point z_r.
Let $\bar{z}_r := (n+|\bar{J}|)d_{max} + 1$, $\underline{z}_r := 0$ and $\Delta := \frac{1}{(2|\bar{J}|x^0_{max})^2}$.
if $\mathcal{F}_{D^{\bar{z}_r}} \neq \emptyset$ **then**
  return $z_r := +\infty$.
else
  while $\bar{z}_r - \underline{z}_r \geq \frac{\Delta}{2}$ **do**
    Let $z := \frac{\underline{z}_r + \bar{z}_r}{2}$.
    if $\mathcal{F}_{D^z} \neq \emptyset$ **then**
      $\underline{z}_r := z$.
    else
      $\bar{z}_r := z$.
    end if
  end while

Let $z_1 := \underline{z}_r - \frac{\Delta}{2}$, $z_0 := \underline{z}_r$ and y^0, y^1 be the optimal solutions of the problems (D^{z_0}) and (D^{z_1}), respectively.
return $z_r := z_0^r$ by (3.38).
end if

3.5. An algorithm to solve the inverse problem ($RIOVLP_1$)

In this subsection, we design an algorithm for the problem $(RIOVLP_1)$. Based on the previous analysis, the problem $(RIOVLP_1)$ can be solved as long as the critical value z^* is determined.

Now we describe the main idea to calculated z^*. Firstly, calculate the left and right break points z_l, z_r by Algorithms 1 and 2. If $k_{z_l}^+ \leq \delta$, then $z^* := z_l$ is the critical value by Case (3.2) in Theorem 5. Similarly, if $k_{z_r}^- \geq \delta$, then $z^* := z_r$ is the critical value by Case (1.1) in Theorem 5. Initialize $\bar{\tau}_0 := (n + |\bar{J}|)d_{max} + 1$ and $\underline{\tau}_0 = -\bar{\tau}_0$. If $k_{\underline{\tau}_0} < \delta$ or $k_{\bar{\tau}_0} > \delta$, then the problem $(RIOVLP_1)$ is infeasible. Now it comes to the case that the problem $(RIOVLP_1)$ is feasible and z_l or z_r is not the critical value, then we can determine an interval $[\underline{\tau}_\kappa, \bar{\tau}_\kappa]$ including z^* by a binary search method which satisfies $k_{\bar{\tau}_\kappa}^- < \delta \leq k_{\underline{\tau}_\kappa}^+$. In the κ-th iteration of the binary search method, if the current value z_κ is a turning coordinate and $k_{z_\kappa}^+ \leq \delta \leq k_{z_\kappa}^-$, then $z^* = z_\kappa$ is the critical value by Case (4.3) in Theorem 5. Otherwise, the binary search method terminates when the length of $|\bar{\tau}_\kappa - \underline{\tau}_\kappa| < \Delta$. In this case, there is only one turning coordinate in the interval $[\underline{\tau}_\kappa, \bar{\tau}_\kappa]$, which is just the critical value.

Next, we discuss the following questions. (Q1) If $z \notin TC$ and $\mathcal{F}_{D^z} \neq \emptyset$, then how to calculate the slope k_z of $\psi(z)$. (Q2) If z is the left break point, then how to calculate the right slope k_z^+ of $\psi(z)$. (Q3) If z is the right break point, then how to calculate the left slope k_z^- of $\psi(z)$. To answer the question (Q1), let y^z be the optimal solution of the problem (D^z). Calculate the optimal solution $(\bar{\pi}^z, \bar{\alpha}^z, \bar{\beta}^z)$ of the problem (P^z) by duality theory of LP. Hence, we have $k_z := \sum_{j \in \bar{J}} (\bar{\beta}_j^z - \bar{\alpha}_j^z) x_j^0$. For the question (Q2), let $z' := z + \frac{\Delta}{2}$, then $z' \notin TC$ and $\mathcal{F}_{D^{z'}} \neq \emptyset$. Hence, we have $k_z^+ := k_{z'}$, where $k_{z'}$ can be calculated similar to the question (Q1). As for the question (Q3), let $z' := z - \frac{\Delta}{2}$, then $z' \notin TC$ and $\mathcal{F}_{D^{z'}} \neq \emptyset$. Hence, we have $k_z^- := k_{z'}$, where $k_{z'}$ can be calculated similar to the question (Q1). For convenience, we give an algorithm to calculate k_z, k_z^+ or k_z^-.

Algorithm 3 k_z=**Slope**(z_l, z_r, z).

Input: The left and right break points z_l, z_r and a value z.
Output: The slope k_z of $\psi(z)$.
if $z = z_l$ **then**
    Let $z' := z + \frac{\Delta}{2}$.
else if $z = z_r$ **then**
    Let $z' := z - \frac{\Delta}{2}$.
else
    Let $z' := z$.
end if
Let $y^{z'}$ be the optimal solution of the problem $(D^{z'})$. Calculate the optimal solution $(\bar{\pi}^{z'}, \bar{\alpha}^{z'}, \bar{\beta}^{z'})$ of the problem $(P^{z'})$ by duality theory of LP.
return $k_z := \sum_{j \in \bar{J}} (\bar{\beta}_j^{z'} - \bar{\alpha}_j^{z'}) x_j^0$.

Next, we give Algorithm 4 to solve the problem $(RIOVLP_1)$.

Algorithm 4 c^*=**RIOVLP**(A, b, c, d, x^0, K).

Input: The coefficient matrix A, the vectors b, c, d, x^0 and a value K.
Output: An optimal solution c^* of the problem $(RIOVLP_1)$.
Calculate z_l :=**LBP**(A, b, c, d, x^0, K) and z_r :=**RBP**(A, b, c, d, x^0, K).
Initialize $\Delta := \frac{1}{(2|\bar{J}|x^0_{max})^2}$, $\delta := c_{\bar{J}} x^0_{\bar{J}} - K$, $z^* := +\infty$ and $\kappa := 0$.
if $z_l = -\infty$ **then**
    Let $\underline{\tau}_0 := -(n + |\bar{J}|) d_{max} - 1$ and $k_{\underline{\tau}_0}$:=**Slope**$(z_l, z_r, \underline{\tau}_0)$.
    if $k_{\underline{\tau}_0} < \delta$ **then**
        The problem $(RIOVLP_1)$ is infeasible and stop.
    end if
else
    Let $\underline{\tau}_0 := z_l$ and $k^+_{\underline{\tau}_0}$:=**Slope**$(z_l, z_r, \underline{\tau}_0)$.
    if $k^+_{\underline{\tau}_0} \leq \delta$ **then**
        Let $z^* := z_l$.
    end if

end if
if $z_r = +\infty$ **then**
  Let $\bar{\tau}_0 := (n + |\bar{J}|)d_{max} + 1$ and $k_{\bar{\tau}_0}$:=**Slope**$(z_l, z_r, \bar{\tau}_0)$.
  if $k_{\bar{\tau}_0} > \delta$ **then**
    The problem $(RIOVLP_1)$ is infeasible and stop.
  end if
else
  Let $\bar{\tau}_0 := z_r$ and $k^-_{\bar{\tau}_0}$:=**Slope**$(z_l, z_r, \bar{\tau}_0)$.
  if $k^-_{\bar{\tau}_0} \geq \delta$ **then**
    Let $z^* := z_r$.
  end if
end if
while $\bar{\tau}_\kappa - \underline{\tau}_\kappa \geq \Delta$ and $z^* = +\infty$ **do**
  Let $z_\kappa := \frac{\bar{\tau}_\kappa + \underline{\tau}_\kappa}{2}$, $z^1_\kappa := z_\kappa - \frac{\Delta}{2}$, $z^2_\kappa := z_\kappa - \frac{\Delta}{4}$, $z^3_\kappa := z_\kappa + \frac{\Delta}{4}$ and $z^4_\kappa := z_\kappa + \frac{\Delta}{2}$.
  Let $k^1_\kappa := \frac{\psi(z_\kappa) - \psi(z^1_\kappa)}{z_\kappa - z^1_\kappa}$, $k^2_\kappa := \frac{\psi(z_\kappa) - \psi(z^2_\kappa)}{z_\kappa - z^2_\kappa}$, $k^3_\kappa := \frac{\psi(z_\kappa) - \psi(z^3_\kappa)}{z_\kappa - z^3_\kappa}$ and $k^4_\kappa := \frac{\psi(z_\kappa) - \psi(z^4_\kappa)}{z_\kappa - z^4_\kappa}$.
  if $k^1_\kappa = k^2_\kappa$ and $k^3_\kappa = k^4_\kappa$ and $k^2_\kappa \neq k^3_\kappa$ **then**
    if $k^4_\kappa \geq \delta$ **then**
      Update $\underline{\tau}_{\kappa+1} := z^4_\kappa$ and $\bar{\tau}_{\kappa+1} := \bar{\tau}_\kappa$.
    else if $k^1_\kappa < \delta$ **then**
      Update $\bar{\tau}_{\kappa+1} := z^1_\kappa$ and $\underline{\tau}_{\kappa+1} := \underline{\tau}_\kappa$.
    else
      Update $z^* := z_\kappa$.
    end if
  else
    Let k_{z_κ} :=**Slope**(z_l, z_r, z_κ).
    if $k_{z_\kappa} \geq \delta$ **then**
      Update $\underline{\tau}_{\kappa+1} := z_\kappa$ and $\bar{\tau}_{\kappa+1} := \bar{\tau}_\kappa$.
    else
      Update $\bar{\tau}_{\kappa+1} := z_\kappa$ and $\underline{\tau}_{\kappa+1} := \underline{\tau}_\kappa$.
    end if

43: **end if**
44: Update $\kappa := \kappa + 1$.
45: **end while**
46: **if** $z^* = +\infty$ **then**
47: Let $k_{\underline{\tau}_\kappa} :=$**Slope**$(z_l, z_r, \underline{\tau}_\kappa)$ and $k_{\bar{\tau}_\kappa} :=$**Slope**$(z_l, z_r, \bar{\tau}^\kappa)$.
48: Calculate $z^* := \frac{\psi(\bar{\tau}_\kappa) - \psi(\underline{\tau}_\kappa) - k_{\bar{\tau}_\kappa}\bar{\tau}_\kappa + k_{\underline{\tau}_\kappa}\underline{\tau}_\kappa}{k_{\underline{\tau}_\kappa} - k_{\bar{\tau}_\kappa}}$.
49: **end if**
50: Let y^* be the optimal solution of the problem (D^{z^*}). Then (y^*, z^*) be the optimal solution of the problem $(DRIOVLP_1)$.
51: Calculate the optimal solution $(\pi^*, \alpha^*, \beta^*)$ of the problem $(RIOVLP_1^2)$ by duality theory of LP.
52: **return** $c^* := c + \alpha^* - \beta^*$.

For convenience, we define $\mathcal{L} = \max\{d_{max}, x^0_{max}, n\}$. Then we can get the time complexity of Algorithm 4.

Theorem 6. *Algorithm 4 can solve the problem (RIOVLP$_1$) by solving the problem (D^z) $O(\log \mathcal{L})$ times at most.*

Proof. The correctness of Algorithm 4 can be obtained by the main idea of the algorithm and Theorem 5. Now we analyze the time complexity. The main computation is in Line 1 and the while loop in Lines 25-45, which are all performed by a binary search method until the length of $|\bar{\tau}_\kappa - \underline{\tau}_\kappa| < \Delta$. We only need to calculate the number of iterations in the while loop. The initial interval length is $|\bar{\tau}_0 - \underline{\tau}_0| = 2(n + |\bar{J}|)d_{max} + 2$ and the interval length will be reduced by at least half in each iteration of a binary search method. Suppose there are t iterations in the while loop in the worst case. Then we have $\big(2(n + |\bar{J}|)d_{max} + 2\big)(\frac{1}{2})^t < \Delta = \frac{1}{(2|\bar{J}|x^0_{max})^2}$, which means $t > 2\log(2|\bar{J}|x^0_{max}) + \log\big(2(n + |\bar{J}|)d_{max} + 2\big)$. Hence, we have $t = \lceil 2\log(2|\bar{J}|x^0_{max}) + \log\big(2(n + \bar{J})d_{max} + 2\big)\rceil \leq \lceil 2\log(2|\bar{J}|x^0_{max}) + \log 3(n + \bar{J})d_{max}\rceil = O(\log \max\{d_{max}, x^0_{max}, n\}) = O(\log \mathcal{L})$. Notice that the Algorithm 4 needs to calculate the problem (D^z) at most five times in each iteration. Therefore, the conclusion holds. □

4. Applications to the Hitchcock and Shortest Path problems

In this section, we apply the previous research methods to the restricted inverse optimal value problems on Hitchcock and shortest path problem under weighted l_1 norm, respectively.

As these two problems can finally be transformed into a minimum cost flow (MCF) problem, we first introduce the problem (MCF) in Ahuja and Orlin (1993).

Let $G(V, E, c, u)$ be a directed network with a cost $c_{ij} > 0$ and a capacity $u_{ij} > 0$ associated with every arc $(i, j) \in E$. We associate with each node $i \in V$ a supply $b(i) > 0$ or a demand $b(i) < 0$. Suppose $\sum_{i \in V} b(i) = 0$, then the problem (MCF) can be stated as follows.

$$
(MCF) \quad \begin{aligned}
\min \quad & \sum_{(i,j)\in E} c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ji} = b(i), i \in V, \\
& 0 \leq x_{ij} \leq u_{ij}, (i, j) \in E.
\end{aligned}
$$

So far, the best strong polynomial time complexity for solving the problem (MCF) is $O(|E| \log |V|(|E| + |V| \log |V|))$ presented by Orlin (1993).

4.1. The restricted inverse optimal value problem on Hitchcock problem under weighted l_1 norm

In this subsection, we study the restricted inverse optimal value problem ($RIOVHC_1$) on Hitchcock problem under weighted l_1 norm.

The Hitchcock problem can be described as follows. We have m sources of some commodity, each with a supply of $a_i > 0$ units, $i = 1, \cdots, m$, and n terminals, each with a demand of $b_j > 0$ units, $j = 1, \cdots, n$. Suppose $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$. There is a unit cost $c_{ij} > 0$ of sending the commodity from source i to terminal j. We aim to satisfy the demands at minimum cost. Hence, the Hitchcock problem (HC) can be stated as follows.

$$
\begin{array}{lll}
 & \min & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\
(HC) & \text{s.t.} & \sum_{j=1}^{n} x_{ij} = a_i, i = 1, \cdots, m, \\
 & & \sum_{i=1}^{m} x_{ij} = b_j, j = 1, \cdots, n, \\
 & & x_{ij} \geq 0.
\end{array}
$$

Let A and c be the coefficient matrix and cost vector of the problem (HC), respectively. The problem ($RIOVHC_1$) can be described as follows. Given a feasible solution x^0 of the problem (HC), a weight vector $d > 0$ and a real number K, we aim to adjust the cost vector c to $\bar{c}$ under weighted l_1 norm such that x^0 becomes an optimal solution of the problem (HC) under $\bar{c}$ and $\bar{c}x^0$ equals K.

Note that A is a unimodular matrix. Suppose Assumption 2 holds. Let $J = \{(i,j)|x^0_{ij} = 0\}$ and $\bar{J} = \{(i,j)|x^0_{ij} > 0\}$. Therefore, if we can solve the following problem (HC-$\breve{D}^z$), then the problem ($RIOVHC_1$) can also be solved by Algorithm 4.

$$
\begin{array}{lll}
 & \min & \sum_{(i,j)\in J\cup\bar{J}} c_{ij}x_{ij} - \sum_{(i,j)\in\bar{J}} c_{ij}x^0_{ij} \\
(HC\text{-}\breve{D}^z) & \text{s.t.} & \sum_{j=1}^{n} x_{ij} = a_i, i = 1, \cdots, m, \\
 & & \sum_{i=1}^{m} x_{ij} = b_j, j = 1, \cdots, n, \\
 & & 0 \leq x_{ij} \leq d_{ij}, (i,j) \in J, \\
 & & (1-z)x^0_{ij} - d_{ij} \leq x_{ij} \leq (1-z)x^0_{ij} + d_{ij}, (i,j) \in \bar{J}.
\end{array}
$$

Let $x'_{ij} = x_{ij}$ for any $(i,j) \in J$, $x'_{ij} = x_{ij} + d_{ij} - (1-z)x^0_{ij}$ for any $(i,j) \in \bar{J}$, $a'_i = a_i - \sum_{j:(i,j)\in\bar{J}} \big((1-z)x^0_{ij} - d_{ij}\big)$, $i = 1, \ldots, n$ and $b'_j =$

$b_j - \sum_{i:(i,j)\in\bar{J}} \big((1-z)x^0_{ij} - d_{ij}\big)$, $j = 1, \ldots, n$. Hence, the problem $(HC\text{-}\breve{D}^z)$ can be turned into the following form.

$$
\begin{aligned}
\min \quad & \sum_{(i,j)\in J\cup\bar{J}} c_{ij}x'_{ij} - \sum_{(i,j)\in\bar{J}} c_{ij}(d_{ij} + zx^0_{ij}) \\
(HC\text{-}\breve{D}^z) \text{ s.t.} \quad & \sum_{j:(i,j)\in J\cup\bar{J}} x'_{ij} = a'_i, i = 1, \cdots, m, \\
& \sum_{i:(i,j)\in J\cup\bar{J}} x'_{ij} = b'_j, j = 1, \cdots, n, \\
& 0 \le x'_{ij} \le d_{ij}, (i,j) \in J, \\
& 0 \le x'_{ij} \le 2d_{ij}, (i,j) \in \bar{J}.
\end{aligned}
$$

Obviously, we have $\sum_{i=1}^m a'_i = \sum_{j=1}^n b'_j$ since $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$. Furthermore, if there exists $a'_i < 0$ or $b'_j < 0$, then the problem $(HC\text{-}\breve{D}^z)$ is infeasible. Otherwise, the problem $(HC\text{-}\breve{D}^z)$ is a Hitchcock problem with upper bound constraints, which can be transformed into an (MCF) problem. Therefore, we can obtain the time complexity of the problem $(RIOVHC_1)$.

Theorem 7. *The restricted inverse optimal value problem $(RIOVHC_1)$ on Hitchcock problem under weighted l_1 norm can be solved by Algorithm 4 in $O\big((m \log n(m + n \log n)) \log \mathcal{L}\big)$ time.*

Next, we present an example to execute Algorithm 4 for the problem $(RIOVHC_1)$.

Example 1. Let v_1, v_2 be the sources of some commodity with a supply of $a_1 := 8, a_2 := 12$ units, and three terminals v_3, v_4, v_5 with a demand of $b_1 := 5, b_2 := 4, b_3 := 11$ units. There is a unit cost $c_{ij} > 0$ of sending the commodity from source i to terminal j as shown in Figure 3. Let $x^0 := (3, 2, 3, 2, 2, 8)$ be a feasible transportation strategy. Let $d := (6, 4, 2, 5, 4, 3)$ and $K := 50$. We aim to adjust the vector c to $\bar{c}$ under weighted l_1 norm such that x^0 becomes an optimal transportation strategy whose cost is just K under $\bar{c}$.

(1) Calculate $J := \emptyset$, $\bar{J} := \{1, 2, 3, 4, 5, 6\}$, $d_{max} := 6$, $x^0_{max} := 8$, $\Delta := \frac{1}{(2|\bar{J}|x^0_{max})^2} := \frac{1}{9216}$ and $\delta := c_{\bar{J}}x^0_{\bar{J}} - K := -6$. (2) Calculate $z_l := -\frac{5}{11}$ and $z_r := \frac{5}{11}$ by Algorithms 1 and 2. (3) Let $\underline{\tau}_0 := z_l$ and $\bar{\tau}_0 := z_r$. Calculate

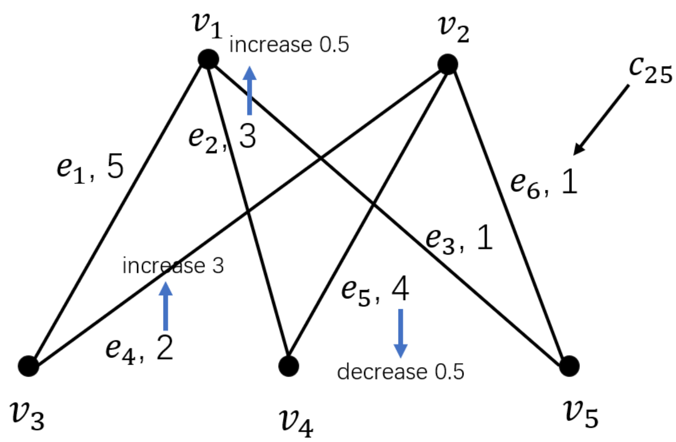

Figure 3: An example of the problem $(RIOVHC_1)$.

$k^+_{\underline{\tau}_0} := 32$ and $k^-_{\bar{\tau}_0} := -17$. Therefore, the critical vale z^* is located in the interval $[\underline{\tau}_0, \bar{\tau}_0]$ since $k^+_{\underline{\tau}_0} \geq \delta > k^-_{\bar{\tau}_0}$. We divide the interval $[\underline{\tau}_0, \bar{\tau}_0]$ by a binary search method. Let $z_0 := \frac{\underline{\tau}_0+\bar{\tau}_0}{2} := 0$. Calculate $z_0^1 := -\frac{1}{18432}$, $z_0^2 := -\frac{1}{36864}$, $z_0^3 := \frac{1}{36864}$, $z_0^4 := \frac{1}{18432}$, and $k_0^1 := k_0^2 := -4$, $k_0^3 := k_0^4 := -8$. Hence, z_0 is a turning coordinate. Notice that $k_0^4 < \delta := -6 \leq k_0^1$. Therefore, z_0 is the critical value and the optimal solution of the problem $(RIOVHC_1)$ is $c^* := (5, \mathbf{3.5}, 1, \mathbf{5}, \mathbf{3.5}, 1)$ as shown in Figure 3. Furthermore, we can draw the graph of function $\psi(z)$ by enumerating z in the interval $[z_l, z_r]$ as shown in Figure 4, where the green dots represent the turning coordinates and the red dot is the critical value.

4.2. The restricted inverse optimal value problem on shortest path problem under weighted l_1 norm

In this subsection, we study the restricted inverse optimal value problem on shortest path problem under weighted l_1 norm $(RIOVSP_1)$.

Let $G = (V, E, c)$ be a directed network, where V, E and c denote the node set, the edge set and the edge cost vector, respectively. Let nodes s and t denote two specified nodes. Suppose the network G does not contain any negative cost cycle, then the $s - t$ shortest path problem can be described

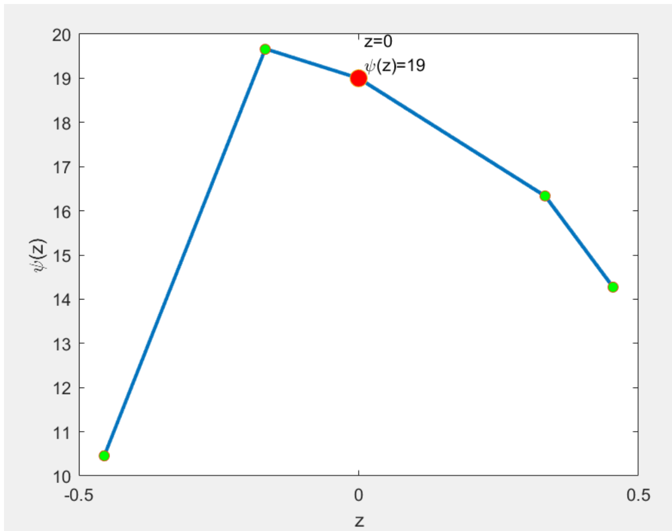

Figure 4: The graph of function $\psi(z)$ in Example 1.

as follows.

$$
\begin{aligned}
& \min && \sum_{(i,j)\in E} c_{ij}x_{ij} \\
(SP)\ & \text{s.t.} && \sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ji} = 1, i = s, \\
& && \sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ji} = 0, i \notin \{s,t\}, \\
& && \sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ji} = -1, i = t, \\
& && x_{ij} \geq 0, (i,j) \in E.
\end{aligned}
$$

The problem $(RIOVSP_1)$ can be described as follows. Let P^0 be a given $s-t$ path and x^0 be the corresponding 0-1 vector whose component 1 indicating the edges on P^0. Let $d > 0$ be the weight vector and K be a real number. We aim to adjust the cost vector c to $\bar{c}$ under weighted l_1 norm such that x^0 becomes a shortest path whose cost equals K on new network $G = (V, E, \bar{c})$.

Obviously, the coefficient matrix of the problem (SP) is unimodular.

Suppose Assumption 2 holds. Let $J = \{(i,j)|x^0_{ij} = 0\}$ and $\bar{J} = \{(i,j)|x^0_{ij} = 1\}$. Then the problem $(RIOVSP_1)$ can also be solved by Algorithm 4 as long as the following problem $(SP\text{-}\breve{D}^z)$ can be solved.

$$
\begin{aligned}
&\min && \sum_{(i,j)\in E} c_{ij}x_{ij} - \sum_{(i,j)\in \bar{J}} c_{ij} \\
&(SP\text{-}\breve{D}^z)\ \text{s.t.} && \sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ji} = 1, i = s, \\
& && \sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ji} = 0, i \notin \{s,t\}, \\
& && \sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ji} = -1, i = t, \\
& && 0 \le x_{ij} \le d_{ij}, (i,j) \in J, \\
& && 1 - z - d_{ij} \le x_{ij} \le 1 - z + d_{ij}, (i,j) \in \bar{J}.
\end{aligned}
$$

Let $x'_{ij} = x_{ij}$ for any $(i,j) \in J$ and $x'_{ij} = x_{ij} - 1 + z + d_{ij}$ for any $(i,j) \in \bar{J}$. For convenience, we assume $P^0 = j_0(s), j_1, \ldots, j_k, j_{k+1}(t)$. Let $b'(i) = 1 - (1 - z - d_{ij_1})$ for $i = s$, $b'(i) = -(1 - z - d_{ij_{h+1}}) + (1 - z - d_{j_{h-1}i})$ for $i = j_h$ and $1 \le h \le k$, and $b'(i) = -1 + (1 - z - d_{j_k i})$ for $i = t$.

Hence, the problem $(SP\text{-}\breve{D}^z)$ can be turned into following form.

$$
\begin{aligned}
&\min && \sum_{(i,j)\in E} c_{ij}x'_{ij} - \sum_{(i,j)\in \bar{J}} c_{ij}(z + d_{ij}) \\
&(SP\text{-}\breve{D}^z)\ \text{s.t.} && \sum_{j:(i,j)\in E} x'_{ij} - \sum_{j:(j,i)\in E} x'_{ji} = b'(i), i = s, \\
& && \sum_{j:(i,j)\in E} x'_{ij} - \sum_{j:(j,i)\in E} x'_{ji} = b'(i), i = j_h, h = 1, \ldots, k, \\
& && \sum_{j:(i,j)\in E} x'_{ij} - \sum_{j:(j,i)\in E} x'_{ji} = 0, i \notin V(P^0), \\
& && \sum_{j:(i,j)\in E} x'_{ij} - \sum_{j:(j,i)\in E} x'_{ji} = b'(i), i = t, \\
& && 0 \le x'_{ij} \le d_{ij}, (i,j) \in J, \\
& && 0 \le x'_{ij} \le 2d_{ij}, (i,j) \in \bar{J}.
\end{aligned}
$$

Notice that for each edge $(j_h, j_{h+1}) \in P_0$ $(h = 0, 1, \cdots, k)$, there is one item $-(1-z-d_{j_h j_{h+1}})$ for $i = j_h$ and one item $+(1-z-d_{j_h j_{h+1}})$ for $i = j_{h+1}$ in $b'(i)$. Then we have $\sum_{i \in V} b'(i) = \sum_{i \in V(P^0)} b'(i) = 0$. Hence, for a given $z \in \mathbb{R}$, the problem $(SP\text{-}\breve{D}^z)$ can be transformed into an (MCF) problem. Therefore, we can obtain the time complexity of the problem $(RIOVSP_1)$.

Theorem 8. *The restricted inverse optimal value problem $(RIOVSP_1)$ on shortest path under weighted l_1 norm can be solved by Algorithm 4 in $O\big((m \log n(m + n \log n)) \log \max\{d_{max}, n\}\big)$ time. Furthermore, the time complexity can be reduced to $O\big((m \log n(m + n \log n)) \log n\big)$ under unit l_1 norm, where $d_{max} = 1$.*

Next, we present an example to execute Algorithm 4 for the problem $(RIOVSP_1)$.

Example 2. Let $G(V, E, c)$ be a directed weighted graph as shown in Figure 5, $P^0 := \{e_2, e_6, e_{10}\}$ (the read edges) be a given $s - t$ path, $c := (2, 3, 7, 8, 5, 6, 4, 9, 1, 10)$, $d := (10, 3, 8, 2, 5, 1, 4, 9, 7, 6)$ and $K := 18$. We aim to adjust the vector c to $\bar{c}$ under weighted l_1 norm such that P^0 becomes a shortest $s - t$ path whose length is just K under $\bar{c}$.

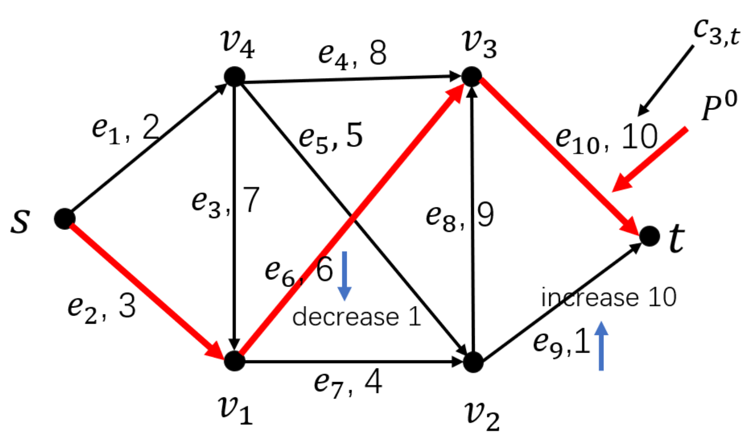

Figure 5: An example of the problem $(RIOVSP_1)$.

(1) Calculate $J := \{1, 3, 4, 5, 7, 8, 9\}$, $\bar{J} := \{2, 6, 10\}$, $d_{max} := 10$, $x^0_{max} := 1$, $\Delta := \frac{1}{(2|\bar{J}|x^0_{max})^2} := \frac{1}{36}$ and $\delta := c_{\bar{J}} x^0_{\bar{J}} - K := 1$. (2) Calculate $z_l := -1$ and

$z_r := 12$ by Algorithms 1 and 2. (3) Let $\underline{\tau}_0 := z_l$ and $\bar{\tau}_0 := z_r$. Calculate $k^+_{\underline{\tau}_0} := 11$ and $k^-_{\bar{\tau}_0} := -13$. Therefore, the critical vale z^* is located in the interval $[\underline{\tau}_0, \bar{\tau}_0]$ since $k^+_{\underline{\tau}_0} \geq \delta > k^-_{\bar{\tau}_0}$. We divide the interval $[\underline{\tau}_0, \bar{\tau}_0]$ by a binary search method. Let $z_0 := \frac{\underline{\tau}_0 + \bar{\tau}_0}{2} := \frac{11}{2}$. Calculate $z_0^1 := \frac{395}{72}$, $z_0^2 := \frac{791}{144}$, $z_0^3 := \frac{793}{144}$, $z_0^4 := \frac{397}{72}$, and $k_0^1 := k_0^2 := k_0^3 := k_0^4 := 11$. Hence, z_0 is not a turning coordinate. Calculate $k_{z_0} := 11$. Hence, $k_{z_0} \geq \delta$ and $\underline{\tau}_1 := z_0 := \frac{11}{2}$, $\bar{\tau}_1 := \bar{\tau}_0 := 12$. We continue to divide the interval $[\underline{\tau}_1, \bar{\tau}_1]$ by a binary search method. After nine iterations, we get the final interval $[\underline{\tau}_9, \bar{\tau}_9] := [\frac{3063}{512}, \frac{769}{128}]$. (4) Calculate $k_{\bar{\tau}_9} := 0$, $k_{\underline{\tau}_9} := 11$, $\psi(\bar{\tau}_9) := 77$ and $\psi(\underline{\tau}_9) := \frac{13902}{181}$. Hence, $z^* := \frac{\psi(\bar{\tau}_9) - \psi(\underline{\tau}_9) - k_{\bar{\tau}_9}\bar{\tau}_9 + k_{\underline{\tau}_9}\underline{\tau}_9}{k_{\underline{\tau}_9} - k_{\bar{\tau}_9}} := 6$. Therefore, the optimal solution of the problem $(RIOVSP_1)$ is $c^* := (2, 3, 7, 8, 5, \mathbf{5}, 4, 9, \mathbf{11}, 10)$ as shown in Figure 5. Furthermore, we can draw the graph of function $\psi(z)$ by enumerating z in the interval $[z_l, z_r]$ as shown in Figure 6, where the green circles represent the turning coordinates and the red dot is the critical value.

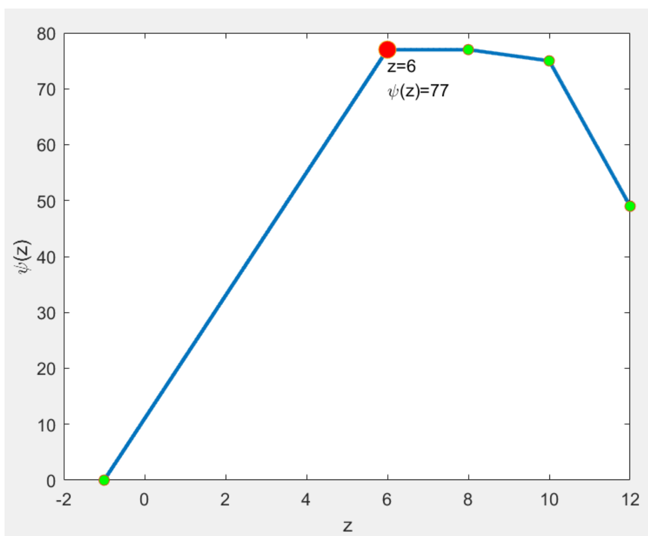

Figure 6: The graph of function $\psi(z)$ in Example 2.

5. Conclusions and further research

In this paper, we mainly study the restricted inverse optimal value problem on (LP) under weighted l_1 norm. Firstly, we construct the mathematical model of the problem $(RIOVLP_1)$ by the dual theories, which is a linear programming problem. Secondly, we introduce a sub-problem (D^z) of the dual inverse problem $(DRIOVLP_1)$ with respect to a given value z which only changes the upper and lower bounds of the variables compared to the original (LP) problem. Thirdly, we design a binary search algorithm to calculate the critical value z^* corresponding the optimal solution (y^*, z^*) of the dual problem $(DRIOVLP_1)$. In each iteration, we need to solve a sub-problem problem (D^z), which can be generally solved by an algorithm for the the original (LP) problem. Finally, we can obtain an optimal solution of the inverse problem $(RIOVLP_1)$ by complementary slackness of LP. The time complexity is $O(T^z \log \mathcal{L})$, where $\mathcal{L} = \max\{d_{max}, x^0_{max}, n\}$ and T^z is the time complexity to solve the sub-problem (D^z). Finally, we apply the research methods to some restricted inverse optimal value problems on Hitchcock and shortest path problems, where the sub-problem (D^z) can be transformed into minimum cost flow problems.

We do not consider the bound constraints on the adjustment amount in this paper, which may render some elements of the adjusted vector $\bar{c}$ too small or too large. In the future, we will study the bounded restricted inverse optimal value problem on LP under weighted l_1 norm and other norms. Furthermore, if the original LP problem is not standard, then we will consider whether our research results can be used to solve the corresponding inverse optimal value problem.

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