FORCING SUBSETS FOR SOME TYPES OF CONVEX SETS IN A GRAPH

Article · February 2018

DOI: 10.17654/DM019010033

Roxanne Arco Anunciado
Caraga State University Cabadbaran City

Advances and Applications in Discrete Mathematics

http://www.pphmj.com
http://dx.doi.org/10.17654/DM019010033
Volume 19, Number 1, 2018, Pages 33-49 **ISSN: 0974-1658**

FORCING SUBSETS FOR SOME TYPES OF CONVEX SETS IN A GRAPH

Roxanne L. Arco and Sergio R. Canoy, Jr.

Department of Mathematics and Statistics
College of Science and Mathematics
Center for Graph Theory, Algebra, and Analysis
Premier Institute of Science and Mathematics
MSU-Iligan Institute of Technology
Iligan City, 9200, Philippines

Abstract

Let G be a connected graph. Given any two vertices u and v of G, the set $I_D[u, v]$ consists of all those vertices lying on a longest u-v path. A set S is a detour convex set if $I_D[u, v] \subseteq S$ for $u, v \in S$. A tolled walk T between distinct vertices u and v of G is a walk of the form $T = [u, w_1, ..., w_k, v]$, where $k \geq 1$, in which w_1 and w_2 are the only neighbors of u and v in T, respectively. The toll interval $T_G(u, v)$ is the set of vertices in G that lie on some u-v walk. A subset $S \subseteq V(G)$ is toll convex (or t-convex) if $T_G(u, v) \subseteq S$ for all $u, v \in S$.

In this paper, we define and study the concepts of detour convexity number, toll convexity number, forcing subset for a maximum detour convex (maximum toll convex) set, and the forcing detour convexity (forcing toll convexity) number of a graph. In particular, we study these concepts in the join and corona of graphs.

Received: July 4, 2017; Accepted: September 5, 2017

2010 Mathematics Subject Classification: 05C12.

Keywords and phrases: detour convex set, toll convex set, forcing detour convexity number, forcing toll convexity number.

1. Introduction

Harary and Nieminen in [7] initiated the study of geodetic convexity in graphs. This type of convexity was further studied in [3-5] where the concept of forcing convexity number of a graph is also introduced and studied.

Recently, two other types of convexity have been considered. Arco and Canoy, Jr. [2] studied detour convexity, characterized the detour convex sets of some graphs and determined their detour convexity numbers. Alcón et al. [1] studied toll convexity, a convexity that uses the concept of a tolled walk. In the latter, the authors have characterized the toll-convex sets of the Cartesian and lexicographic products of some graphs and introduced other invariants arising from toll convexity such as the toll number and toll hull number of a graph. Toll convexity in graphs is also studied by Gologranc and Repolusk in [6].

Let G be a (simple) connected graph and let $u, v \in V(G)$. The *detour distance* $D(u, v)$ of u and v is the length of a longest u-v path in G. A u-v path of length $D(u, v)$ is called a *u-v detour*. If u and v are two distinct non-adjacent vertices in G, then a *tolled walk* T between u and v in G is a sequence of vertices of the form $T = [u, w_1, ..., w_k, v]$, where $k \geq 1$, which enjoys the following three conditions:

- $w_i w_{i+1} \in E(G)$ for all i,
- $uw_i \in E(G)$ if and only if $i = 1$,
- $vw_i \in E(G)$ if and only if $i = k$.

The set $I_D^G[u, v]$ or simply $I_D[u, v]$ (resp. $T_G(u, v)$) consists of all vertices lying on some u-v detour (resp. u-v tolled walk) of G. For $S \subseteq V(G)$, $I_D[S] = \bigcup_{u, v \in S} I_D[u, v]$. A subset S of $V(G)$ is a *detour convex* (resp. *toll convex* or *t-convex*) set if $I_D[u, v] \subseteq S$ (resp. $T_G(u, v) \subseteq S$) for every $u, v \in S$. The *detour convexity number* $con_D(G)$ (resp. *toll convexity* *number* $con_T(G)$) of G is the maximum cardinality of a proper detour convex (resp. proper *t*-convex) set of G. Any detour convex (resp. *t*-convex) set S of G with $|S| = con_D(G)$ (resp. $|S| = con_T(G)$) is called a *maximum detour convex set* or con_D*-set* (resp. *maximum t-convex set* or con_D*-set*) of G. A subset Q of a con_D-set (resp. con_T-set) S of G is called a *forcing subset* for S if S is the unique con_D-set (resp. con_T-set) containing Q. The *forcing detour convexity number* $fcon_D(S)$ (resp. *forcing toll convexity number* $fcon_T(S)$) of a con_D-set (resp. con_T-set) S of G is the minimum cardinality of a forcing subset for S. The *forcing detour convexity number* $fcon_D(S)$ (resp. *forcing toll convexity number* $fcon_T(G)$) of G is the minimum forcing detour convexity number (resp. minimum forcing toll convexity number) among all con_D-sets (resp. con_T-sets) of G.

In this paper, the authors deal with the concepts of detour convexity number, toll convexity number, forcing subset for a maximum detour convex (maximum toll convex) set, and the forcing detour convexity (forcing toll convexity) numbers in the join and corona of graphs.

2. Forcing Subsets for a con_D-set of a Graph

This section deals with the detour convex sets and the forcing subsets for the con_D-sets of some graphs. In particular, these types of sets are investigated in the join and corona of graphs.

Remark 2.1. Let G be a connected graph.

(i) If S is a con_D-set of G, then S is a forcing subset for itself. In particular, $fcon_D(G) \leq con_D(G)$.

(ii) If G has a unique con_D-set S, then the empty set $\varnothing$ is a forcing subset for S. In this case, $fcon_D(G) = 0$.

Theorem 2.2. *Let G be a connected graph of order n. Then* $0 \leq fcon_D(G) \leq n-1$. *Furthermore,*

(i) $fcon_D(G) = 0$ *if and only if G has a unique* con_D*-set; and*

(ii) $fcon_D(G) = 1$ *if and only if G does not have a unique* con_D*-set but some vertex of G belongs to exactly one* con_D*-set.*

Proof. The first statement follows directly from Remark 2.1.

(i) Suppose that $fcon_D(G) = 0$. Then there exists a con_D-set S of G with $fcon_D(S) = 0$. This means that $\varnothing$ is the minimum forcing subset for S and S is the unique con_D-set of G containing $\varnothing$. Hence, S is the unique con_D-set of G.

Conversely, assume that S is a unique con_D-set of G. Then by Remark 2.1(ii), $fcon_D(G) = 0$.

(ii) Suppose that $fcon_D(G) = 1$. Then there exists a con_D-set S of G having the set $\{v\}$ as its minimum forcing subset for some $v \in V(G) \cap S$. Since $\varnothing$ is not the minimum forcing subset for S, G has another con_D-set, say S', and $v \notin S'$.

The converse is easy. □

A vertex v of G is a *detour extreme vertex* of G if it is an initial or terminal vertex of any detour containing v. The set of all detour extreme vertices of G is denoted by $Ex_D(G)$.

Note that if $Ex_D(G) \neq \varnothing$, then $con_D(G) = |V(G)| - 1$. In particular, $S = V(G) \setminus \{x\}$ is a con_D-set of G for each $x \in Ex_D(G)$.

Theorem 2.3. *Let G be a connected graph with k detour extreme vertices* $(k \geq 1)$. *Then* $fcon_D(G) = k - 1$.

Proof. Let G be a connected graph with k detour extreme vertices, where $k \geq 1$ and let S be a con_D-set of G. Then there exists $x \in Ex_D(G)$ such that $S = V(G)\backslash\{x\}$. If $k = 1$, then S is the unique con_D-set of G. Thus, $fcon_D(G) = 0$ by Theorem 2.2. Now, assume that $k \geq 2$ and let $T \subseteq S$. If there exists $y \in Ex_D(G)\backslash\{x\}$ that is not in T, then $T \subseteq S' = V(G)\backslash\{y\}$. Since S' is a con_D-set of G different from S, it follows that T is not a forcing subset for S. Hence, $T \subseteq S$ is a forcing subset for S if $Ex_D(G)\backslash\{x\} \subseteq T$. Since $Ex_D(G)\backslash\{x\}$ is a forcing subset for S, $fcon_D(S) = |Ex_D(G)\backslash\{x\}| = k - 1$. Since every con_D-set of G is similar to S, we have $fcon_D(G) = k - 1$.

□

Corollary 2.4. *If G is a Hamiltonian graph, then* $fcon_D(G) = 1$.

The converse of Corollary 2.4 is not true as the next result shows.

Theorem 2.5. $fcon_D(K_{m,n}) = 1$ *for* $m = n = 1$ *or* $m, n \geq 2$.

Proof. It can easily be verified that every singleton is a con_D-set of G. Thus, by Theorem 2.2(ii), $fcon_D(G) = 1$. □

Theorem 2.6. *Let* $J_k = \{1, 2, ..., k\}$ *and let* $C_1, C_2, ..., C_k$ *be the components of a graph G. Then* $fcon_D(K_1 + G) = |\mathcal{R}| - 1$, *where* $\mathcal{R} = \{r \in J_k : C_r$ *is a component of G of least order*$\}$.

Proof. The con_D-sets of G are the sets of the form $V(K_1 + G)\backslash V(C_m)$, where $m \in \mathcal{R}$. If $|\mathcal{R}| = 1$, say $i \in \mathcal{R}$, then $S = V(K_1 + G)\backslash V(C_i)$ is the unique con_D-set of G. Thus, by Theorem 2.2, $fcon_D(G) = 0$. Now, suppose that $|\mathcal{R}| \geq 2$. Let $S_1 = V(K_1 + G)\backslash V(C_r)(r \in \mathcal{R})$ and let $T \subseteq S_1$. If there exists $j \in \mathcal{R}\backslash\{r\}$ such that $T \cap V(C_j) = \varnothing$, then $T \subseteq S_j = V(K_1 + G)\backslash V(C_j)$. This implies that T is not a forcing subset for S_1. Thus, if T is a forcing subset for S_1, then $T \cap V(C_t) \neq \varnothing$ for each $t \in \mathcal{R}\backslash\{r\}$. Pick

$x_t \in V(C_t)$ for each $t \in \mathcal{R}\backslash\{r\}$ and consider $T_0 = \{x_t : t \in \mathcal{R}\backslash\{r\}\}$. Clearly, T_0 is a minimum forcing subset for S_1. Hence, $fcon_D(S_1) = |\mathcal{R}| - 1$. Since every con_D-set of $K_1 + G$ is similar to S_1, it follows that $fcon_D(K_1 + G)$ $= |\mathcal{R}| - 1$. □

Recall that the *corona* of two graphs G and H, denoted by $G \circ H$, is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then forming the join $\langle v \rangle + H^v = v + H^v$ for every vertex v of G, where H^v denotes a copy of H for each vertex v.

The next result is found in [2].

Theorem 2.7. *Let G be a connected graph and let H be any graph with k components. A non-empty subset C of $V(G \circ H)$ is a detour convex set of $G \circ H$ if and only if one of the following holds*:

(i) $C = V(G \circ H)$;

(ii) $C = \{u\}$ *for some* $u \in V(G \circ H)$;

(iii) $C \subseteq V(G)$, *where C is a detour convex set of G*; *or*

(iv) $C = S \bigcup T$ *such that S is a detour convex set of $V(G)$ and* $T = \bigcup_{v \in S'} \bigcup_{i_v \in \mathcal{K}_v} V(C_v^{i_v})$, *where* $S' \subseteq S$, $C_v^{i_v}$ *is a component of* H^v *and* $\mathcal{K}_v \subseteq \mathcal{K} = \{1, 2, ..., k\}$ *for each* $v \in S'$.

Theorem 2.8. *Let G be a connected graph of order $m \geq 2$ and let H be any graph with components C_i, where $i \in J_k = \{1, 2, ..., k\}$. Then* $fcon_D(G \circ H) = m|\mathcal{R}| - 1$, *where* $\mathcal{R} = \{r \in J_k : C_r$ *is a component of H of least order*$\}$.

Proof. Let $V(G) = \{v_1, v_2, ..., v_m\}$. Then v_i is a cut-vertex of $G \circ H$ for every $i \in I_m = \{1, 2, ..., m\}$. Now, for each $i \in I_m$, let $J_{i_{k(i)}} = \{i_1, i_2, ...,$

$i_{k(i)}\}$ and let $C_{i_1}, C_{i_2}, ..., C_{i_{k(i)}}$ be the components of $(G \circ H) - v_i$. Suppose further that

$$\zeta = \min\{|V(C_{i_q})| : 1 \leq i \leq m, 1 \leq q \leq k(i)\}.$$

For each $i \in I_m$, let $\mathcal{R}_i = \{r \in J_{i_{k(i)}} : C_r$ is a component of $(G \circ H) - v_i$ with $|V(C_r)| = \zeta\}$. Clearly, $|\mathcal{R}_i| = |\mathcal{R}|$ for all $i \in I_m$. Next, let C be a con_D-set of $G \circ H$. By Theorem 2.7, there exists a $v_i \in V(G)$ such that $C = V(G \circ H)\setminus V(C_r)$ for some $r \in \mathcal{R}_i$. Let D be a non-empty forcing subset for C. Suppose that there exists $v_j \in V(G)\setminus\{v_i\}$ and $q \in \mathcal{R}_i$ such that $D \cap V(C_q) = \varnothing$. Then $D \subseteq C^* = V(G \circ H)\setminus V(C_q)$. Since C^* is a con_D-set of $G \circ H$ different from C, it follows that D is not a forcing subset for C, a contradiction. Thus, $D \cap V(C_q) \neq \varnothing$ for all $q \in \mathcal{R}_j \setminus \{r\}$ for each $j \in I_m$. This implies that

$$\begin{aligned} fcon_D(C) &= \sum_{j \in I_m \setminus \{i\}} |\mathcal{R}_j| + (|\mathcal{R}_i| - 1) \\ &= (m-1)|\mathcal{R}| + |\mathcal{R}| - 1 \\ &= m|\mathcal{R}| - 1. \end{aligned}$$

Since every other con_D-set of $G \circ H$ is similar to C, $fcon_D(G) = m|\mathcal{R}| - 1$. □

3. Forcing Subsets for a con_T-set of a Graph

A vertex x from a t-convex set S is said to be a *toll extreme vertex* of S if $S \setminus \{x\}$ is t-convex. Throughout this section, $Ex_T(G) = \{x \in V(G) : x$ is a toll extreme vertex of $V(G)\}$.

Remark 3.1. Let G be a connected graph of order n.

(i) A vertex x is a toll extreme vertex of G if and only if $V(G)\setminus\{x\}$ is a t-convex set of G. Furthermore, if $Ex_T(G) \neq \emptyset$, then $con_T(G) = n-1$.

(ii) If S is a con_T-set of G, then S is a forcing subset for itself. In particular, $fcon_T(G) \leq con_T(G)$.

(iii) If G has a unique con_T-set S, then the empty set $\emptyset$ is a forcing subset for S. In this case, $fcon_T(G) = 0$.

Theorem 3.2. *Let G be a connected graph with* $|Ex_T(G)| = k \geq 1$. *Then* $fcon_T(G) = k-1$.

Proof. Let G be a connected graph and suppose that $|Ex_T(G)| = k \geq 1$. Let S be a con_T-set of G. Then, by Remark 3.1(i), there exists $x \in Ex_T(G)$ such that $S = V(G)\setminus\{x\}$. If $k = 1$, then S is the unique con_T-set of G. Thus, $fcon_T(G) = 0$ by Remark 3.1(iii). Next, assume that $k \geq 2$ and let $T \subseteq S$. If there exists $y \in Ex_T(G)\setminus\{x\}$ that is not in T, then $T \subseteq S' = V(G)\setminus\{y\}$, where S' is a con_T-set of G. It follows that T is not a forcing subset for S. Hence, $T \subseteq S$ is a forcing subset for S if $Ex_T(G)\setminus\{x\} \subseteq T$. Since $Ex_T(G)\setminus\{x\}$ is a forcing subset for S, $fcon_T(S) = |Ex_T(G)\setminus\{x\}| = k-1$. Since every con_T-set of G is similar to S, we have $fcon_T(G) = k-1$. □

Observe that in a complete graph, every vertex is a toll extreme vertex. Thus, the next result is immediate from this observation.

Theorem 3.3. *For* $n \geq 1$, $con_T(K_n) = fcon_T(K_n) = n-1$.

The next result characterizes the t-convex sets in the join of non-complete graphs G and H.

Theorem 3.4. *Let G and H be non-complete graphs. A non-empty proper subset S of* $V(G+H)$ *is a t-convex set of* $G+H$ *if and only if* $S = S_G \cup S_H$, *where* $\langle S_G \rangle$ *and* $\langle S_H \rangle$ *are cliques of G and H, respectively* (S_G *or* S_H *may be empty*).

Proof. Let S be a non-empty t-convex set of $G + H$ and let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. Consider the following cases:

Case 1. $S \subseteq V(G)$ or $S \subseteq V(H)$

Assume that $S \subseteq V(G)$. Then $S = S_G$ and $S_H = \emptyset$. Let $u, v \in S$ such that $u \neq v$. Suppose that $uv \notin E(G)$. Then the walk $T = [u, w, v]$ is a tolled walk between u and v in $G + H$ for all $w \in V(H)$. It follows that $V(H) \subseteq T_{G+H}(u, v) \subseteq S$, a contradiction. Hence, $uv \in E(G)$ for every $u, v \in S$. Therefore, $\langle S \rangle = \langle S_G \rangle$ is a clique of G. Similarly, $S_G = \emptyset$ and $\langle S \rangle = \langle S_H \rangle$ is a clique of H if $S \subseteq V(H)$.

Case 2. $S_G \neq \emptyset$ and $S_H \neq \emptyset$

Suppose that $S_G = V(G)$. Since $S \neq V(G + H)$, $S_H \neq V(H)$. Now, since G is not complete, there exist distinct vertices $a, b \in V(G)$ such that $ab \notin E(G)$. Pick any $z \in V(H) \setminus S_H$. Then $T = [a, z, b]$ is a tolled walk in $G + H$ with $z \notin S$. Hence, S is not a t-convex set, a contradiction. Thus, $S \neq V(G)$. Similarly, $S_H \neq V(H)$. If $\langle S_G \rangle$ is not complete, then $V(H) \subseteq S$. Hence $S_H = V(H)$, a contradiction. Therefore, $\langle S_G \rangle$ is a clique of G. Similarly, $\langle S_H \rangle$ is a clique of H.

The converse is clear because every clique of $G + H$ is a t-convex set of $G + H$. □

The next result is a consequence of Theorem 3.4.

Corollary 3.5. *Let G and H be non-complete graphs. Then $S \subset V(G + H)$ is a con_T-set of $G + H$ if and only if $S = S_G \cup S_H$, where $\langle S_G \rangle$ and $\langle S_H \rangle$ are maximum cliques of G and H, respectively. Moreover, $con_T(G + H) = \omega(G) + \omega(H) = \omega(G + H)$.*

Let G be a connected graph and let K_r be a maximum clique of G. A subset S of $V(K_r)$ is a *c-forcing subset* for $V(K_r)$ if K_r is the only maximum clique of G such that $S \subseteq V(K_r)$. The *forcing clique number* of K_r is given by

$$fcn(K_r) = \min\{|S| : S \text{ is a } c\text{-forcing subset for } V(K_r)\}.$$

The *forcing clique number* of G is given by

$$fcn(G) = \min\{fcn(K_r) : K_r \text{ is a maximum clique of } G\}.$$

Remark 3.6. Let G be a connected graph. Then $fcn(G) = 0$ if and only if G has a unique maximum clique. If the con_T-sets of G induce maximum cliques of G, then $fcon_T(S) = fcn(\langle S\rangle)$ for every con_T-set S of G. Therefore $fcon_T(G) = fcn(G)$.

Theorem 3.7. *Let G and H be non-complete graphs. Then*

$$fcon_T(G+H) = fcn(G) + fcn(H).$$

Proof. Let G and H be non-complete graphs. Assume that $\langle S_G\rangle$ and $\langle S_H\rangle$ are maximum cliques of G and H, respectively, such that $fcn(G) = fcn(\langle S_G\rangle)$ and $fcn(H) = fcn(\langle S_H\rangle)$. Let C_G and C_H be forcing subsets for S_G and S_H, respectively, such that $fcn(\langle S_G\rangle) = |C_G|$ and $fcn(\langle S_H\rangle) = |C_H|$. By Theorem 3.4, $S = S_G \cup S_H$ is a con_T-set of $G+H$. Let $D = C_G \cup C_H$. Suppose that $D \subseteq S'$ for some con_T-set S' of $G+H$ distinct from S. Let $S' = S'_G \cup S'_H$, where $\langle S'_G\rangle$ and $\langle S'_H\rangle$ are maximum cliques of G and H, respectively. It follows that $C_G \subseteq S'_G$ and $C_H \subseteq S'_H$. Since $S' \neq S$, either $S'_G \neq S_G$ or $S'_H \neq S_H$. If $S'_G \neq S_G$, then $C_G \subseteq S_G \cap S'_G$ implies that C_G is not a forcing subset for S_G. This gives a contradiction. A similar case happens if $S'_H \neq S_H$. Hence, D is a forcing subset for S. Consequently, $fcon_T(G+H) \leq fcon_T(S) \leq |D| = fcn(G) + fcn(H)$.

Next, let S^* be a con_T-set of $G+H$ with $fcon_T(G+H) = fcon_T(S^*)$ $= fcn(\langle S^*\rangle)$. By Theorem 3.4, $S^* = S^*_G \cup S^*_H$, where $\langle S^*_G\rangle$ and $\langle S^*_H\rangle$ are maximum cliques of G and H, respectively. Suppose that C is a forcing subset for S^* such that $fcn(\langle S^* \rangle) = |C|$. Assume further that $C = A \cup B$, where $A \subseteq S_G^*$ and $B \subseteq S_H^*$. If $A \subseteq T_G$ for some maximum clique $\langle T_G \rangle$ of G and $T_G \neq S_G^*$, then $T_G \cup S_H^*$ is a con_T-set of $G + H$ and $C \subseteq T_G \cup S_H^*$, contrary to the assumption that C is a forcing subset for S^*. Similarly, B is a forcing subset for S_H^*. Therefore $fcon_T(G + H) = |C| = |A| + |B| \geq fcn(G) + fcn(H)$.

Accordingly, $fcon_T(G + H) = fcn(G) + fcn(H)$. □

Corollary 3.8. *Let G and H be non-complete graphs. Then*

(i) $fcon_T(G + H) = 0$ *if and only if G and H have unique maximum cliques; and*

(ii) $fcon_T(G + H) = 1$ *if and only if either G has a unique maximum clique and* $fcn(H) = 1$ *or H has a unique maximum clique and* $fcn(G) = 1$.

Corollary 3.9. *Let G and H be non-complete graphs. Then* $fcon_T(G + H) = 2$ *if and only if one of the following conditions holds:*

(i) $fcn(G) = 1$ *and* $fcn(H) = 1$;

(ii) $fcn(G) = 2$ *and H has a unique maximum clique; or*

(iii) $fcn(H) = 2$ *and G has a unique maximum clique.*

Theorem 3.10. *Let G be any graph and let H be a complete graph. Then a non-empty proper subset S of* $V(G + H)$ *is a t-convex set of* $G + H$ *if and only if*

(i) $S = S_G \cup S_H$, *where* $\langle S_G \rangle$ *is a clique of G and* $S_H \subseteq H$ (S_G *or* S_H *may be empty); or*

(ii) $S = C \cup V(H)$, *where C is a proper t-convex set of G such that* $\langle C \rangle$ *is not a clique of G.*

Proof. Let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. Consider the following cases:

Case 1. $S \subseteq V(G)$ or $S \subseteq V(H)$

We may assume that $S \subseteq V(G)$. Then $S = S_G$. Assume further that $\langle S_G \rangle$ is not a clique of G. Then there exist $u, v \in S$, $u \neq v$, such that $uv \notin E(G)$. This implies that $V(H) \subseteq S$, a contradiction. Thus, $\langle S \rangle = \langle S_G \rangle$ is a clique of G.

Case 2. $S_G \neq \varnothing$ and $S_H \neq \varnothing$

If G is complete, then $\langle S_G \rangle$ is a clique of G. Assume now that G is not complete and suppose that $S_G = V(G)$. Since $S \neq V(G+H)$, $S_H \neq V(H)$. Pick non-adjacent vertices x, y of S_G and let $z \in V(H) \setminus S_H$. Then $z \in T_{G+H}(x, y) \subseteq S$, a contradiction. Thus, $S_G \neq V(G)$. If $\langle S_G \rangle$ is complete, then (i) holds. If $\langle S_G \rangle$ is not complete, then there exist $u, v \in S_G$ with $uv \notin E(G)$. This would imply that $V(H) \subseteq T_{G+H}(u, v) \subseteq S$. Hence, $S_H = V(H)$. Let $C = S_G$ and suppose that C is not a t-convex set of G. Then there exist $a, b \in C$ such that there is an $x \in T_G(a, b) \setminus S_G$. Let $P(a, b)$ be a tolled walk in G containing x. Then $P(a, b)$ is a tolled walk in $G + H$. Hence, S is not a t-convex set of $G + H$, a contradiction. It follows that $C = S_G$ is a t-convex set of G, showing that (ii) holds.

The converse is clear. □

Corollary 3.11. *Let G be any graph and let H be a complete graph of order n. Then S is a con_T-set of $G + H$ if and only if $S = C \cup V(H)$ for some con_T-set C of G. Moreover,*

$$con_T(G + H) = con_T(G) + n.$$

In particular, if every con_T-set of G is a clique, then

$$con_T(G + H) = \begin{cases} \omega(G) + n, & \text{if } G \text{ is not complete}, \\ \omega(G) + n - 1, & \text{otherwise.} \end{cases}$$

Theorem 3.12. *Let G be a graph and let H be a complete graph. Then*

$$fcon_T(G + H) = fcon_T(G).$$

Proof. Let S be a con_T-set of $G + H$ and let T be a forcing subset for S such that $fcon_T(G + H) = fcon_T(S) = |T|$. Then, by Corollary 3.11, $S = C \cup V(H)$, where C is a con_T-set of G. Let $T = T_1 \cup T_2$, where $T_1 \subseteq C$ and $T_2 \subseteq V(H)$. Suppose that $T_1 \subseteq C'$ for some con_T-set C' of G with $C' \neq C$. Then $T \subseteq S' = C' \cup V(H)$. Contrary to the assumption that T is a forcing subset for S. Thus, T_1 is a forcing subset for C. Hence, T_1 is also a forcing subset for S. Therefore $fcon_T(G + H) = fcon_T(S) = |T| \geq |T_1| \geq fcon_T(G)$.

Now, let Q be a con_T-set of G and let P be a forcing subset for Q such that $fcon_T(G) = fcon_T(Q) = |P|$. Then $S^* = Q \cup V(H)$ is a con_T-set of $G + H$ by Corollary 3.11. Clearly, P is also a forcing subset for S^*. Therefore $fcon_T(G + H) \leq fcon_T(S^*) \leq |P| = fcon_T(G)$.

Accordingly, $fcon_T(G + H) = fcon_T(G)$. □

The next result characterizes the t-convex sets in the corona of two graphs G and H.

Theorem 3.13. *Let G be a nontrivial connected graph and let H be any graph. Then C is a t-convex set of* $G \circ H$ *if and only if* $C = S_v$, *where* $\langle S_v \rangle$ *is a complete subgraph of* H^v *for some* $v \in V(G)$ *or* $C = A \cup \left(\bigcup_{v \in A} D_v \right)$, *where A is a non-empty t-convex set of G and* D_v *is a t-convex set of* H^v *for each* $v \in A$.

Proof. Suppose that C is a t-convex set of $G \circ H$. Let $A = C \cap V(G)$. Suppose that $A = \varnothing$ and let $x \in C$ and $v \in V(G)$ such that $x \in S_v = C \cap V(H^v)$. Let $u \in V(G) \setminus \{v\}$ and set $S_u = C \cap V(H^u)$. Suppose that $S_u \neq \varnothing$, say $y \in S_u$. Let $P(u, v) = [u_1, u_2, ..., u_n]$, where $u = u_1$ and $v = u_n$, be a u-v geodesic in G. Then $P(x, y) = [x, u_1, u_2, ..., u_n, y]$ is an x-y tolled walk in $G \circ H$. Since $u, v \notin C$, it follows that C is not a t-convex set of $G \circ H$, a contradiction. Thus, $S_u = \varnothing$ for all $u \in V(G) \setminus \{v\}$. Moreover, since $v \notin A$, $\langle S_v \rangle$ must be a complete subgraph of H^v, where $C = S_v$.

Next, suppose that $A \neq \varnothing$. Since every tolled walk in G is a tolled walk in $G \circ H$, it follows that A is a t-convex set of G. Let $v \in A$ and let $D_v = C \cap V(H^u)$. Again, it is a routine to show that $D_w = C \cap V(H^w) = \varnothing$ for all $w \in V(G) \setminus A$. Hence, $C = A \cup \left(\bigcup_{v \in A} D_v \right)$, where D_v is necessarily a t-convex set of H^v for each $v \in A$.

The converse is clear. □

Theorem 3.14. *Let G be a nontrivial connected graph of order m and let H be any graph of order n. Then $con_T(G \circ H) = m + (m-1)n + con_T(H)$.*

Proof. Let $A = V(G)$ and let $v \in V(G)$. Set $D_w = V(H^w)$ for each $w \in A \setminus \{v\}$ and let D_v be a con_T-set of H^v. By Theorem 3.13, $C = A \cup \left(\bigcup_{x \in A} D_x \right)$ is a t-convex set of $G \circ H$. Hence,

$$con_T(G \circ H) \geq |C| = m + (m-1)n + con_T(H).$$

Next, suppose that $C_0 = A_0 \cup \left(\bigcup_{y \in A_0} D'_y \right)$ be a con_T-set of $G \circ H$. If $A_0 = V(G)$, then there exists $z \in A_0$ such that D'_z is a con_T-set of H^z.

Hence,

$$con_T(G \circ H) = |C_0| = m + \sum_{y \in A_0 \setminus \{z\}} D'_y + |D'_z|$$

$$\leq m + (m-1)n + con_T(H).$$

If $A_0 \neq V(G)$, then $con_T(G \circ H) = |C_0| \leq m + (m-1)n \leq m + (m-1)n + con_T(H)$. Therefore $con_T(G \circ H) = m + (m-1)n + con_T(H)$. □

Theorem 3.15. *Let G be a nontrivial connected graph of order m and let H be any graph. Then* $fcon_T(G \circ H) \leq m(fcon_T(H)+1)-1$. *Moreover, if H has a unique* con_T*-set, then* $fcon_T(G \circ H) = m-1$.

Proof. Let R be a con_T-set of H and let S be a forcing subset for R such that $fcon_T(H) = fcon_T(R) = |S|$. For each $v \in V(G)$, let R_v be a con_T-set of H^v and let S_v be a forcing subset for R_v such that $\langle R_v \rangle \cong \langle R \rangle$ and $\langle S_v \rangle \cong \langle S \rangle$. By Theorem 3.14, $C = V(G) \cup \left(\bigcup_{v \in V(G) \setminus \{w\}} V(H^v) \right) \cup R_w$ is a con_T-set of $G \circ H$. For each $v \in V(G) \setminus \{w\}$, pick $z_v \in V(H^v) \setminus R_v$. Let $Q_w = S_w \cup \left(\bigcup_{v \in V(G) \setminus \{w\}} (S_v \cup \{z_v\}) \right)$. Assume that $C' = V(G) \cup \left(\bigcup_{v \in V(G) \setminus \{y\}} V(H^v) \right) \cup D_y$ is a con_T-set of $G \circ H$ with $C' \neq C$. Suppose first that $y = w$. Then $D_y \neq R_w$. Since S_w is a forcing subset for R_w, $S_w \not\subseteq D_y$. Next, suppose that $y \neq w$. If $D_y = R_y$, then $z_y \in Q_w \setminus D_y$. If $D_y \neq R_y$, then $S_y \not\subseteq D_y$ because S_y is a forcing subset for R_y. Thus, $Q_w \not\subseteq C'$. This implies that Q_w is a forcing subset for C. Hence,

$$fcon_T(G \circ H) \leq fcon_T(C) \leq |Q_w| = fcon_T(H) + (m-1)(fcon_T(H)+1)$$

$$= m(fcon_T(H)+1) - 1.$$

Let C_0 be a con_T-set of $G \circ H$ and let Q_0 be a forcing subset for

C_0 such that $fcon_T(G \circ H) = fcon_T(C_0) = |Q_0|$. Then $C_0 = V(G) \cup \left(\bigcup_{v \in V(G)\setminus\{z\}} V(H^v)\right) \cup R_z$ for some $z \in V(G)$, by Theorem 3.14 (since H has a unique con_T-set). For each $v \in V(G)\setminus\{z\}$, let $N_v = Q_0 \cap V(H^v)$. If $N_v \cap (V(H^v)\setminus R_v) = \varnothing$, then

$$Q_0 \subseteq C^* = V(G) \cup \left(\bigcup_{x \in V(G)\setminus\{v\}} V(H^x)\right) \cup R_v,$$

where C^* is a con_T-set of $G \circ H$ different from C_0. This gives a contradiction since Q_0 is a forcing subset for C_0. Hence, $N_v \cap (V(H^v)\setminus R_v) \neq \varnothing$ for every $v \in V(G)\setminus\{z\}$. Thus, $fcon_T(G \circ H) = |Q_0| \geq m - 1$.

Accordingly, $fcon_T(G \circ H) = m - 1$. □

Acknowledgements

This research is funded by the Department of Science and Technology-Accelerated-Science and Technology Human Resource Development Program (DOST-ASTHEDP), and the MSU-Iligan Institute of Technology, Iligan City, Philippines.

References

[1] L. Alcón, B. Brešar, T. Gologranc, M. Gutierrez, T. Kraner, I. Peterin and A. Tepeh, Toll convexity, European J. Combin. 46 (2015), 161-175.

[2] R. Arco and S. Canoy, Jr., Detour convexity in graphs, J. Anal. Appl. 15(2) (2017), 117-131.

[3] S. Canoy, Jr. and L. Decasa, A note on the forcing convexity number of graphs, Congressus Numerantium 172 (2005), 33-41.

[4] G. Chartrand, C. Wall and P. Zhang, The convexity number of a graph, Graphs and Combinatorics 18 (2002), 209-217.

[5] G. Chartrand and P. Zhang, The forcing convexity number of a graph, Czech. Math. J. 51(126) (2001), 847-858.

[6] T. Gologranc and P. Repolusk, Toll number of the Cartesian and the lexicographic product of graphs, arXiv:1608.07390v1 [math.CO] (2016).

[7] F. Harary and J. Nieminen, Convexity in graphs, J. Differential Geometry 16 (1981), 185-190.