A BIVARIATE OPTIMAL REPLACEMENT POLICY FOR A REPAIRABLE SYSTEM WITH NONN REPAIR TIMES UNDER ALPHA SERIES PROCESS

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ABSTRACT

In this paper a bivariate optimal replacement policy for a repairable system consisting of one unit and a single repairman with NONN repair times under Alpha series processe is studied . Assume that the system after repair is not as good as new. Under this assumption, a bivariate optimal replacement policy (T, N), and we consider a replacement policy T is adapted by which the system will be replaced whenever the working age of the system reaches T; another replacement policy N is the number of failures of the system with NONN repair times of the system is studied. Furthermore, The successive survival times of the system after repair form a decreasing Alpha series process or a renewal process while the consecutive repair times of the system consitute an increasing Alpha series process or a renewal process. Our problem is to determine an optimal replacement policy $(T, N)^*$, such that the long run average cost rate of the system is minimized. The explicit expression of the long- run average cost is derived, the corresponding optimal replacement policy can be determined analytically.

Keywords: Alpha Series Process, Renewal Process, Replacement Policy, Convolution Renewal Reward Theorem.

1. INTRODUCTION

A typical maintenance and replacement model may be described in the following way, a system (or a component) is subject to random failure;upon failure, the system is either repaired or replaced by a new and identical one. Moreover, we assume that the repair of a failure system will yield a functioning system which is "as good as new" under this assumption, the succssive survival times of the system after repair or replacement consitute a renewal process. Also, the repair times are assumed to be negligible or non negligible (NONN). It is not true that the repair times always negligible. The repair times usually depend on the type of failure occurred at specific moment in the system. To over come this, Thangaraj and Rizwan (2001) have introduced probability for different types of failure of a repairable system and called it NONN repair times.Thus it is reasonable to assume this alternative repair model.

On the other hand , the replacement policy is either T based on the working age of the system or N based on the failure number of the system. We know that the working age of the system is actually a cumulative time when the system is in working state. If the system has to make replacement or repair decision based on the working age, it requies extra effort to measure and record the actual working time of the system. Clearly, this is more trouble from the view of system engineering management.

In this article, a simple repairable system with NONN repair is studied. By using the Alpha series process repair model. It was introducd by Braun (2005), we consider a optimal replacement policy T based on the system age under which the system is replaced when the system age reaches T. The system age is calender age of the system, another replacement policy N is the number of failures of the system with NONN repair times of the system which is very easy to measure from the point of view of engineering management. Our problem is to determine an optimal replacement policy $(T, \text{N})^*$, such that the average cost rate of the system is minimized.

2. ALPHA SERIES PROCESS REPAIR MODEL

In order to introduce the Alpha series process repair model for a deteriorating system with NONN repair times. We first give some definitions and assumptions.

2.1. Definition

Now we state the definitions of NONN repair times , stochastic order and Alpha series process as follows.

Definition 1

If a repair to a system after failure is done in negligible or non-negligible time, then it will be called a model with NONN repair times.

In this case, whenever the system fails, two possibilities may arise: either , the repair takes Negligible time with probability p; or Non-Negligible time with probability $1 - p$.

Definition 2

Given two random variables X and Yif $P(X > a) \geq P(Y > a)$ for all real a,

Then X is called stochastically larger than Y or Y is stochastically less than X. This is denoted by $X \geq_{st} Y$ or $Y \leq_{st} X$ (See e.g. Ross [5] for reference.)

Definition 3

Given a stochastic process $\{Z_n, n = 1,2 \dots\}$ if for all n, $Z_n \leq_{st} (\geq_{st})Z_{n+1}$, then $\{Z_n, n = 1,2 \dots\}$ is called a stochastically increasing (decreasing) process.

Definition 4

Let $\{X_n, n = 1,2,3 \ldots\}$ be a sequence of independent, non negative random variables. If the distribution function of Y_n is given by $H(n^\alpha x)$ for $n = 1,2,3 \ldots$ where α is a real number. Then $\{X_n, n = 1,2,3 \ldots\}$ is called an Alpha series process. (See e.g., Braun (2005))

According to Definition 2.4, we have

(i) Given a Alpha series process$\{X_n, n = 1,2,3 \ldots\}$

(a) If $\alpha > 0$ then $\{X_n, n = 1,2,3 \ldots\}$ is a stochastically decreasing sequence.

(b) If $\alpha < 0$ then $\{X_n, n = 1,2,3 \ldots\}$ is a stochastically increasing sequence.

(c) If $\alpha = 0$ then $\{X_n, n = 1,2,3 \ldots\}$ is a renewal process.

(ii) Let$E(x) = \gamma, Var(x) = \sigma^2$ then$E(X_n) = \frac{\gamma}{n^\alpha}$. See e.g. Braun (2005)

2.2. Model Assumptions

We consider a maintenance and replacement model and make the following assumptions.

Assumption 2.1. At time $t = 0$, a new system is used. Whenever the system fails, it it will be repaired. The system will be replaced by an identical new one, some times later.

Assumption 2.2. The system after failure cannot be repaired immediately, and the system after repair is not "as good as new". The time interval between the completion of the $(n-1)th$ repair and the completion of the $n - th$ repair of the system is defined as the $n - th$ cycle of the system, for $n = 1,23, \ldots$.

Let X_n, Y_n be the random variables with represent the operating time, the repair time of the system in the $-th$ cycle, for $n = 1,2,3, \ldots$. Respectively. According to definition 4, ,

we have $F_n(x) = F(n^{\alpha_1}x)$, $G_n(y) = G(n^{\alpha_2}y)$ where $\alpha_1 > 0$ and $\alpha_2 \leq 0$, for $n = 1,23, \ldots$

Moreover, assuming that $E(X_1) = \gamma > 0, E(Y_1) = \mu > 0$.

Let X_1 be the first operating time, for $n \geq 2$, Let $X_n$be the operating time of the system after$(n-1)st$ repair. Then the distribution function of $X(n)$is $F(n^{\alpha_1}x)$for $n = 1,2,3,\ldots$ where $\alpha_1 > 0$ is a real number .That is the Successive survival times $\{X_n, n = 1,23, \ldots\}$form an decreasing α -series process . $E(X_n) = \frac{\gamma}{n^{\alpha_1}}$.

Let ξ_n be the NONN repair time after the $n - th$ failure $n \geq 1$

By assumption, $E(\xi_n) = E(Y_n)P(Y_n > 0) + 1\, P(Y_n = 0)$

$$= \frac{\mu}{n^{\alpha_2}}(1 - p) + p$$

and Z be the replacement time.

Assumption 2.3. The survival times (X_i); the NONN repair times (ξ_i) and the replacement time $Z, i = 1,23, \ldots$ are independent.

Assumption 2.4. Let r be the reward rate per unit time of the system when it is operating and c be the repair cost per unit time of the system. Assume further that the replacement cost

comprises of two parts: one part is the basic replacement cost R and the other part is the cost proportional to the length of replacement time Z at rate c_p.

Assumption 2.5. The replacement policy T is adapted under which the system will be replaced whenever its working age reaches T.

3. THE REPLACEMENT POLICY T WITH NONN REPAIR TIMES.

Our problem is to determine an optimal replacement policy T^* such that the long-run average cost per unit time is minimized.

Let T_1 be the first replacement time and in general for $n \geq 2$, let T_n be the time between the $(n-1) - st$ replacement and $n - th$ replacement. Thus the sequence $T_1, T_2, \ldots\ldots\ldots$ constitutes a renewal process. Further , a cycle is completed, if a replacement is done. By the renewal reward theorem, the long-run average cost per unit time under the replacement policy $-T$ for a deteriorating system with NONN repair times, is given by

$$C(T) = \frac{the\ expected\ cost\ incurred\ in\ a\ cycle}{the\ expected\ length\ of\ a\ cycle}$$

$$= \frac{cE\left(\sum_{k=1}^{\eta-1} \xi_k\right) + R + c_\rho E(Z) - rE\left(\sum_{k=1}^{\eta} X_k\right)}{E\left(\sum_{k=1}^{\eta} X_k\right) + E\left(\sum_{k=1}^{\eta-1} \xi_k\right) + E(Z)} \tag{1}$$

Where η is a random variable denoting the number of failures in time T. Since η is also a stopping time with respect to the $\sigma -$ fields $\{\sigma(X_1, X_2, \ldots\ldots X_n)\}$, we have

$$E\left(\sum_{k=1}^{\eta} X_k\right) = E[E(X_\eta|\eta)]$$

$$= \sum_{n=1}^{\infty}[\sum_{k=1}^{n} E(X_k)]P(K = k)$$

$$= \sum_{n=1}^{\infty}\left[\sum_{k=1}^{n} \frac{\gamma}{n^{\alpha_1}}\right] F_n(T) - F_{n+1}(T)$$

$$=$$

$$\gamma \sum_{n=1}^{\infty}\left(\frac{1}{n^{\alpha_1}}\right) F_n(T) \tag{2}$$

Where $F_n(.)$ Is the n-fold convolution of $F(.)$ with itself and

Consider $E\left(\sum_{k=1}^{\eta-1} \xi_k\right) = E\left[E\left(\sum_{k=1}^{\eta-1} E(\xi_k|\eta)\right)\right]$

$$= \sum_{n=1}^{\infty}[\sum_{k=1}^{n-1} E(\xi_k)]P(K = k - 1)$$

$$= \sum_{n=1}^{\infty}\left[\sum_{k=1}^{n} \frac{\mu}{n^{\alpha_2}}(1 - p) + p\right] G_{n-1}(T) - G_n(T)$$

$$= \mu \sum_{n=1}^{\infty}\left(\frac{1}{n^{\alpha_2}}(1 - p) +$$

$$p\right) G_n(T) \tag{3}$$

Where $G_n(.)$ is the n-fold convolution of $G(.)$ with itself

Using equation (2) and (3) , equation (1) becomes

$$C(T) = \frac{c\sum_{n=1}^{\infty}\left(\frac{\mu}{n^{\alpha_2}}(1-p)+p\right)G_n(T)+R+c_\rho\tau-r\gamma\sum_{n=1}^{\infty}\left(\frac{1}{n^{\alpha_1}}\right)F_n(T)}{\gamma\sum_{n=1}^{\infty}\left(\frac{1}{n^{\alpha_1}}\right)F_n(T)+\sum_{n=1}^{\infty}\left(\frac{\mu}{n^{\alpha_2}}(1-p)+p\right)G_n(T)+\tau} \tag{4}$$

Where $E(z) = \tau$. Further

$$C(T) = \frac{(c+r)\sum_{n=1}^{\infty}\left(\frac{\mu}{n^{\alpha_2}}(1-p)+p\right)G_n(T)+R_1}{\gamma\sum_{n=1}^{\infty}\left(\frac{1}{n^{\alpha_1}}\right)F_n(T)+\sum_{n=1}^{\infty}\left(\frac{\mu}{n^{\alpha_2}}(1-p)+p\right)G_n(T)+\tau}$$
$$-r \tag{5}$$

Where $R_1 = R + (c_\rho + r)\tau$.

Let $A_n = \frac{\gamma}{n^{\alpha_1}}$ and $B_n = \frac{\mu}{n^{\alpha_2}}(1-p)+p$

Then $C(T)$ can be rewritten as

$$C(T) = \frac{(c+r)\sum_{n=1}^{\infty}B_n\,G_n(T)+R_1}{\sum_{n=1}^{\infty}A_n\,F_n(T)+\sum_{n=1}^{\infty}B_n\,G_n(T)+\tau}$$
$$-r \tag{6}$$

We observe here that $\sum_{n=1}^{\infty}B_n\,G_n(T)$ and $\sum_{n=1}^{\infty}A_n\,F_n(T)$ converge absolutely. It follows from Mertens theorem [Rudin (1976)] that

$$\sum_{n=1}^{\infty}B_n\,G_n(T)\sum_{n=1}^{\infty}A_n\,F_n(T) = \sum_{n=1}^{\infty}\sum_{k=1}^{n}G_k\,(T)\,B_kF_{n-k}(T)A_{n-k}$$

Since both the series are uniformly convergent, term by term differentiation is applicable. We now differentiate (6) with respect to T and equating it to zero, and simplify it to obtain

$$C'(T) = \frac{\begin{bmatrix}(\sum_{n=1}^{\infty}A_n\,F_n(T)+\sum_{n=1}^{\infty}B_n\,G_n(T)+\tau)\left((c+r)\sum_{n=1}^{\infty}B_n\,G_n'(T)\right)-\\((c+r)\sum_{n=1}^{\infty}B_n\,G_n(T)+R_1)\left(\sum_{n=1}^{\infty}A_n\,F_n{}'(T)+\sum_{n=1}^{\infty}B_n\,G_n{}'(T)\right)\end{bmatrix}}{(\sum_{n=1}^{\infty}A_n\,F_n(T)+\sum_{n=1}^{\infty}B_n\,G_n(T)+\tau)^2}$$

$$C'(T)$$
$$=\frac{\begin{bmatrix}(c+r)\sum_{n=1}^{\infty}\sum_{k=1}^{n}B_k\,A_{n-k}\left(G_k'(T)F_{n-k}(T)-G_k(T)F_{n-k}{}'(T)\right)+\tau(c+r)\left(\sum_{n=1}^{\infty}B_n\,G_n'(T)\right)\\-R_1\left(\sum_{n=1}^{\infty}B_n\,G_n{}'(T)+\sum_{n=1}^{\infty}A_n\,F_n{}'(T)\right)\end{bmatrix}}{(\sum_{n=1}^{\infty}A_n\,F_n(T)+\sum_{n=1}^{\infty}B_n\,G_n(T)+\tau)^2}$$

On equating $C'(T)$ to zero, we obtain

$$(c + r) \sum_{n=1}^{\infty} \sum_{k=1}^{n} B_k A_{n-k} \left(G_k'(T) F_{n-k}(T) - G_k(T) F_{n-k}'(T) \right) + \tau(c + r) \left(\sum_{n=1}^{\infty} B_n G_n'(T) \right)$$

$$-R_1 \left(\sum_{n=1}^{\infty} B_n G_n'(T) + \sum_{n=1}^{\infty} A_n F_n'(T) \right) = 0 \qquad (7)$$

If $C''(T) > 0$, then we must have

$$(c + r) \sum_{n=1}^{\infty} \sum_{k=1}^{n} B_k A_{n-k} \left(G_k''(T) F_{n-k}(T) - G_k(T) F_{n-k}''(T) \right) + \tau(c + r) \left(\sum_{n=1}^{\infty} B_n G_n''(T) \right)$$

$$-R_1 \left(\sum_{n=1}^{\infty} B_n G_n''(T) + \sum_{n=1}^{\infty} A_n F_n''(T) \right) > 0 \qquad (8)$$

For $C(T)$ to attain its minimum, $C'(T) = 0$ and $C''(T) > 0$. Thus (7) gives T^* for which $C(T^*)$ is minimum. Summarizing the above facts, we have the following result.

Theorem 3.1 The long run average cost per unit time , $C(T)$ given by (4) for the monotone process alternative repair model of a repairable system under T policy with NONN repair times is minimum, if (7) and (8) hold.

Remarks.

(i) When $p = 0$, that is , when the repair times are non-negligible, equation (4) reduces to

$$C(T) = \frac{c \sum_{n=1}^{\infty} \left(\frac{\mu}{n^{\alpha_2}} \right) G_n(T) + R + c_\rho \tau - r\gamma \sum_{n=1}^{\infty} \left(\frac{1}{n^{\alpha_1}} \right) F_n(T)}{\gamma \sum_{n=1}^{\infty} \left(\frac{1}{n^{\alpha_1}} \right) F_n(T) + \sum_{n=1}^{\infty} \left(\frac{\mu}{n^{\alpha_2}} \right) G_n(T) + \tau}$$

(i) When $p = 1$, that is , when the repair times are negligible, equation (4) reduces to

$$C(T) = \frac{c \sum_{n=1}^{\infty} G_n(T) + R + c_\rho \tau - r\gamma \sum_{n=1}^{\infty} \left(\frac{1}{n^{\alpha_1}} \right) F_n(T)}{\gamma \sum_{n=1}^{\infty} \left(\frac{1}{n^{\alpha_1}} \right) F_n(T) + \sum_{n=1}^{\infty} G_n(T) + \tau}$$

4. THE REPLACEMENT POLICY N WITH NONN REPAIR TIMES

In this section, we study the alternative repair model if the previous section for the maintenance problem of a repairable system and we use the $N - policy$ with NONN repair times. Under the replacement policy N, the problem is to determine an optimal N^* such that the long-run average cost per unit time is minimized. We make the following assumptions.

Assumptions 4.1 to 4.6 are the same as assumptions 2.2 to 2.6.

4.7. A replacement policy N with NONN repair times is adopted. By applying the replacement policy$N$, the system will be replaced by an identical new one at the time following the $N - th$ failure. The replacement time is a random variable Z with $E(Z) = \tau$.

By the renewal reward theorem, the long-run average cost per unit time under the replacement policy $- N$ for a multistate degenerative system with NONN repair times, is given by

$$C(N) = \frac{The\ expected\ cost\ incurred\ in\ a\ cycle}{The\ expected\ length\ of\ a\ cycle}$$

$$= \frac{CE(\sum_{n=1}^{N-1}\xi_n) + R + C_\rho E(Z) - rE(\sum_{n=1}^{N}X_n)}{E(\sum_{n=1}^{N}X_n) + E(\sum_{n=1}^{N-1}\xi_n) + E(Z)}$$

$$= \frac{(C+r)\sum_{n=1}^{N-1}E\xi_n + R_1}{\sum_{n=1}^{N-1}E\xi_n + \sum_{n=1}^{N}EX_n + \tau} - r \qquad (1)$$

where

$$R_1 = (C_\rho + r)\tau + R$$
$$= A(N) - r$$

where

$$A(N) = \frac{(C+r)\sum_{n=1}^{N-1}E\xi_n + R_1}{\sum_{n=1}^{N-1}E\xi_n + \sum_{n=1}^{N}EX_n + \tau}$$

In order to minimize $C(N)$, we define $A(N)$ and note that minimizing $C(N)$ is equivalent to minimizing $A(N)$.

Now our aim is to determine an optimal policy N^* for minimizing $C(N)$ explicitly. Since $C(N) = A(N) - r$. We can minimize $A(N)$ for determining an optimal policy N^* instead of minimizing $C(N)$. Now we study the difference between $A(N+1)$ and $A(N)$. We have

$$A(N+1) - A(N) = \frac{(C+r)\sum_{n=1}^{N}E\xi_n + R_1}{\sum_{n=1}^{N}E\xi_n + \sum_{n=1}^{N+1}EX_n + \tau} - \frac{(C+r)\sum_{n=1}^{N-1}E\xi_n + R_1}{\sum_{n=1}^{N-1}E\xi_n + \sum_{n=1}^{N}EX_n + \tau}$$

$$= \frac{(C+r)E\xi_N\sum_{n=1}^{N}EX_n - (C+r)EX_{N+1}\sum_{n=1}^{N-1}E\xi_n - R_1E\xi_N - R_1EX_{N+1}}{(\sum_{n=1}^{N}E\xi_n + \sum_{n=1}^{N+1}EX_n + \tau)(\sum_{n=1}^{N-1}E\xi_n + \sum_{n=1}^{N}EX_n + \tau)}$$

$$= \frac{(C+r)[E\xi_N(\sum_{n=1}^{N}EX_n + \tau) - EX_{N+1}\sum_{n=1}^{N-1}E\xi_n] - R_1(E\xi_N + EX_{N+1})}{(\sum_{n=1}^{N}E\xi_n + \sum_{n=1}^{N+1}EX_n + \tau)(\sum_{n=1}^{N-1}E\xi_n + \sum_{n=1}^{N}EX_n + \tau)}$$

According to the numerator of $A(N+1) - A(N)$, we structure an auxillary function.

$$B(N) = \frac{(C+r)[E\xi_N(\sum_{n=1}^{N}EX_n + \tau) - EX_{N+1}\sum_{n=1}^{N-1}E\xi_n]}{R_1(E\xi_N + EX_{N+1})}$$

Because the denominator of $A(N + 1) - A(N)$ is always positive. It is clear that the sign of $A(N + 1) - A(N)$ is the same as the sign of its numerator. So we have the following lemma.

Lemma4.1. $A(N + 1) > (=, <)A(N) \Leftrightarrow B(N) > (=, <)1$

Lemma (4.1) shows that the monotonicity of $A(N)$ is determined by the value of $B(N)$.

Lemma 4.2. $B(N)$ is non-decreasing in N.

$$B(N + 1) - B(N)$$
$$= \left(\frac{(C + r)[E\xi_{N+1}(\sum_{n=1}^{N+1} EX_n + \tau) - EX_{N+2} \sum_{n=1}^{N} E\xi_n]}{R_1(E\xi_{N+1} + EX_{N+2})} \right)$$
$$- \left(\frac{(C + r)[E\xi_N(\sum_{n=1}^{N} EX_n + \tau) - EX_{N+1} \sum_{n=1}^{N-1} E\xi_n]}{R_1(E\xi_N + EX_{N+1})} \right)$$

$$= \left[G(N) \left(\begin{array}{c} E\xi_{N+1}EX_{N+1} \sum_{n=1}^{N+1} EX_n + \tau EX_{N+1}E\xi_{N+1} + EX_{N+1}E\xi_{N+1} \sum_{n=1}^{N} E\xi_n \\ -EX_{N+2}E\xi_N \sum_{n=1}^{N} E\xi_n - EX_{N+2}E\xi_N \sum_{n=1}^{N+1} EX_n - \tau EX_{N+2}E\xi_N \end{array} \right) \right]$$

where

$$G(N) = \frac{(C + r)}{R_1(E\xi_{N+1} + EX_{N+2})(E\xi_N + EX_{N+1})}$$

$$= G(N) \left[E\xi_{N+1}EX_{N+1} \left(\sum_{n=1}^{N+1} EX_n + \tau + \sum_{n=1}^{N} E\xi_n \right) - EX_{N+2}E\xi_N \left(\sum_{n=1}^{N+1} EX_n + \tau + \sum_{n=1}^{N} E\xi_n \right) \right]$$

$$= G(N) \left(\left(\sum_{n=1}^{N+1} EX_n + \tau + \sum_{n=1}^{N} E\xi_n \right) (E\xi_{N+1}EX_{N+1} - EX_{N+2}E\xi_N) \right)$$

$$= \frac{(C + r)\left((\sum_{n=1}^{N+1} EX_n + \tau + \sum_{n=1}^{N} E\xi_n)(E\xi_{N+1}EX_{N+1} - EX_{N+2}E\xi_N) \right)}{R_1(E\xi_{N+1} + EX_{N+2})(E\xi_N + EX_{N+1})}$$

Since X_n is stochastically non-increasing in n and ξ_n is stochastically non decreasing in n, we have $EX_{N+1} \geq EX_{N+2}$ and $E\xi_{N+1} \geq E\xi_N$, respectively. Therefore $E\xi_{N+1}EX_{N+1} - EX_{N+2}E\xi_N \geq 0$, then the inequality above holds.

Theorem 4.1. The optimal replacement policy N^* can be determined by

$$N^* = \min \{N/B(N) \geq 1\}$$

Proof. According to lemmas 4.1 and 4.2, an analytic expression for an optimal policy for minimizing $A(N)$ is obtained through the study of $B(N)$.

Remarks 4.1

$$E(\xi_n) = \frac{\mu}{n^{\alpha_2}}(1-p) + p$$

$$E(X_n) = \frac{\gamma}{n^{\alpha_1}}$$

$$C(N) = \frac{(C+r)\sum_{n=1}^{N-1} E\xi_n + R_1}{\sum_{n=1}^{N-1} E\xi_n + \sum_{n=1}^{N} EX_n + \tau} - r$$

$$C(N) = \frac{(C+r)\sum_{n=1}^{N-1}\left(\frac{\mu}{n^{\alpha_2}}(1-p) + p\right) + R_1}{\sum_{n=1}^{N-1}\left(\frac{\mu}{n^{\alpha_2}}(1-p) + p\right) + \sum_{n=1}^{N}\frac{\gamma}{n^{\alpha_1}} + \tau} - r$$

$$B(N) = \frac{(C+r)[E\xi_N(\sum_{n=1}^{N} EX_n + \tau) - EX_{N+1}\sum_{n=1}^{N-1} E\xi_n]}{R_1(E\xi_N + EX_{N+1})}$$

$$B(N) = \frac{(C+r)\left[\left(\frac{\mu}{N^{\alpha_2}}(1-p) + p\right)\left(\sum_{n=1}^{N}\frac{\gamma}{n^{\alpha_1}} + \tau\right) - \frac{\gamma}{(N+1)^{\alpha_1}}\sum_{n=1}^{N-1}\left(\frac{\mu}{n^{\alpha_2}}(1-p) + p\right)\right]}{(C_\rho + r)\tau + R\left(\left(\frac{\mu}{N^{\alpha_2}}(1-p) + p\right) + \left(\frac{\gamma}{(N+1)^{\alpha_1}}\right)\right)}$$

Remarks 4.2

(i) When $p = 0$, that is , when the repair times are non-negligible, $C(N)$ reduces to

$$C(N) = \frac{(C+r)\sum_{n=1}^{N-1}\left(\frac{\mu}{n^{\alpha_2}}\right) + R_1}{\sum_{n=1}^{N-1}\left(\frac{\mu}{n^{\alpha_2}}\right) + \sum_{n=1}^{N}\frac{\gamma}{n^{\alpha_1}} + \tau} - r$$

(ii) When $p = 1$, that is , when the repair times are negligible, $C(N)$ reduces to

$$C(N) = \frac{(C+r)(N-1) + R_1}{(N-1) + \sum_{n=1}^{N}\frac{\gamma}{n^{\alpha_1}} + \tau} - r$$

Remark 4.3. If $N = 1$ then $B(N)$ reduces to

$$B(1) = \frac{(C+r)}{(C_\rho + r)\tau + R}\left[\frac{\left[(\mu(1-p) + p)(\gamma + \tau) - \frac{\gamma\mu}{(2)^{\alpha_1}}\right]}{\left((\mu(1-p) + p) + \left(\frac{\gamma}{(2)^{\alpha_1}}\right)\right)}\right]$$

If $B(1) \geq 1$, then $N^* = 1$, this means that an optimal replacement policy is to be replace the system immediately, whenever it fails.

If $B(\infty)$ exist and $B(\infty) \leq 1$, then $N^* = \infty$. This means the optimal policy is to continually repair the system as it ages without ever replacing it.

5. THE BIVARIATE POLICY (T, N) WITH NONN REPAIR TIMES.

In this section, we study a bivariate policy (T, N) with NONN repair times **u**nder which the system is replaced at working age T or at the time following the N-th failure whichever occurs first. The problem is to choose a optimal replacemet policy $(T, \text{N})^*$ such that the long run average cost rate of the system is minimized. We make the following assumptions.

Assumptions 5.1. At time $t = 0$, a new system is put in to field use.Whenever the system fails, it will be repaired. The system will be replaced by an identical new one,some time later.

Assumptions 5.2. The system state at time t, denoted by $S(t)$ is

$$S(t) = \begin{cases} 0, & \text{if the system is working at time t} \\ i, & \text{if the system is in the } i^{th} \text{ type of failure state at time t}, \quad i1,2,3 \dots .. k \end{cases}$$

Thus the state space is $\Omega = \{0,1,2 \dots \dots k\}$ if the system fails, the with probability p, the system will be in state i, $i = 1,2, \dots \dots k$ and $\sum_{i=1}^{k} p_i = 1$.

Assumptions 5.3. Let X_1 be the first operatig time. For $n \geq 2$, Let X_n be the operating time of the system after $(n-1) - st$ repair. Let Y_n be the repair time after the $n - th$ failure ad Z be the replacement time. Now, denote the time of the $n - th$ failure by t_n.

Assume that $P(X_1 \leq t) = U(t)$ and $P(X_2 \leq t | S(t_1) = i) = U(a_i t), \quad i = 1,2 \dots .. k$

In general, for $j = 1,2 \dots .. n - 1; \ i_j = 1,2 \dots k$

$$P\left(X_n \leq t | S(t_1) = i_1, \dots \dots S(t_{n-1}) = i_{n-1}\right) = U(a_{i_1}, \dots \dots \dots \dots a_{i_{n-1}} t))$$

Where $1 \leq a_1 \leq a_2 \leq \cdots \dots .. \leq a_k$

Similarly, assume that and $P(Y_1 \leq t | S(t_1) = i) = V(b_i t), \quad i = 1,2 \dots .. k$

In general, for $j = 1,2 \dots .. n; \ i_j = 1,2 \dots k$

$$P\left((Y_n \leq) | S(t_1) = i_1, \dots \dots S(t_n) = i_n\right) = V(b_{i_1}, \dots \dots \dots \dots b_{i_n} t)$$

Where $1 \geq b_1 \geq b_2 \geq \cdots \dots .. \geq b_k > 0$

Assumptions 5.4. The working age of the system at time T is the cumulative life-time given

by $T(t) = \begin{cases} t - M_n, & L_n + M_n \leq t < L_{n+1} + M_n \\ L_{n+1}, & L_{n+1} + M_n \leq t < L_{n+1} + M_{n+1} \end{cases}$

where $L_n = \sum_{i=1}^{n} X_i$ and $M_n = \sum_{i=1}^{n} Y_i$ and $M_0 = L_0 = 0$.

Assumptions 5.5. Let r be the reward rate per unit time of the system when it is operating and c be the repair cost per unit time of the system. Assume further that the replacement cost comprises of two parts: one part is the basic replacement cost R and the other part is the cost proportional to the length of replacement time Z at rate $c_{p.}$

Assumptions 5.6. The replacement policy (T, N) is used.

Let T_1 be the first replacement time and in general for $n \geq 2$, Let T_n be the time between the $(n-1)-st$ replacement and the $n-$th replacement. Then the sequence $\{T_{n,} n1,2 \dots \}$ forms a renewal process. Therefore, the inter arrival times between two consecutive replacement is a renewal cycle.

Let $C(T, \mathrm{N})$ be the long-run average cost per unit time under the replacement policy (T, N)
for a repairable system with NONN repair times, is given by

$$C(T, \mathrm{N}) = \frac{the\ expected\ cost\ incurred\ in\ a\ cycle}{the\ expected\ length\ of\ a\ cycle}$$

$$= \frac{E\{(C\sum_{n=1}^{\eta}\xi_n - r\tau)\chi_{(L_N>T)}\} + C_\rho E(Z) + E\{(C\sum_{n=1}^{N-1}\xi_n - r\sum_{n=1}^{N}X_n)\chi_{(L_N\leq T)}\} + R}{E(W)}$$

(1)

Where η is a random variable denoting the number of failures before the working age of the system reaches T, W is the length of the cycle and $\chi(.)$ denotes the indicator function. Therefore $\eta = 0,1,2 \dots \dots .. N-1$.

The length of the cycle under the replacement policy (T, N) is

$$W = \left(T + \sum_{n=1}^{\eta}\xi_n\right)\chi_{(L_N>T)} + \left(\sum_{n=1}^{N-1}\xi_n + \sum_{n=1}^{N}X_n\right)\chi_{(L_N\leq T)} + Z$$

Where $\eta = 0,1,2 \dots \dots \dots N-1$ is the number of failures before the working age of the system exceeds T.

Now,

$$E\left[\left(\sum_{n=1}^{N}X_n\right)\chi_{(L_N\leq T)}\right] = E\left\{E\left[\left(\left(\sum_{n=1}^{N}X_n\right)\chi_{(L_N\leq T)}\Big|L_N\right)\right]\right\}$$

$$= \int_0^T E\left(\sum_{n=1}^{N}X_n\,|L_n\right)\,dF_N(u)$$

$$= \int_0^T u\,d\,F_N(u) \tag{2}$$

and $E\left[\left(\sum_{n=1}^{N-1}\xi_n\right)\chi_{(L_N\leq T)}\right] = E\left\{E\left[\left(\left(\sum_{n=1}^{N-1}\xi_n\right)\chi_{(L_N\leq T)}\Big|L_N\right)\right]\right\}$

$$= \int_0^T E\left(\left(\sum_{n=1}^{N-1}\xi_n\,|L_N = u\right)\right)\,dF_N(u)$$

$$= \int_0^T \left(\sum_{n=1}^{N-1}E(\xi_n)\right)\,d\,F_N(u)$$

$$= \int_0^T \sum_{n=1}^{N-1} \left(\frac{\mu}{n^{\alpha_2}} (1-p) + p \right) dF_N(u)$$

$$= \sum_{n=1}^{N-1} \left(\frac{\mu}{n^{\alpha_2}} (1-p) + p \right) \int_0^T dF_N(u)$$

$$= \sum_{n=1}^{N-1} \left(\frac{\mu}{n^{\alpha_2}} (1-p) + p \right) F_N(T) \tag{3}$$

Further

$$E(W) = E\left[\left(T + \sum_{n=1}^{\eta} \xi_n \right) \chi_{(L_N > T)} \right] + E\left[\left(\sum_{n=1}^{N-1} \xi_n + \sum_{n=1}^{N} X_n \right) \chi_{(L_N \leq T)} \right] + E(Z)$$

$$= E[T \chi_{(L_N > T)}] + E\left[\left(\sum_{n=1}^{\eta} \xi_n \right) \chi_{(L_N > T)} \right] + E\left\{ E\left[\left(\sum_{n=1}^{N} X_n + \right. \right. \right.$$

$$\left. \left. \left. \sum_{n=1}^{N-1} \xi_n \right) \chi_{\left((L_N \leq T) | L_N \right)} \right] \right\} + E(Z)$$

$$= T\overline{F}_N(T) + E\left[\sum_{n=1}^{N-1} \xi_n \chi_{(L_i < T < L_N)} \right] + \int_0^T u \, dF_N(u) + \sum_{n=1}^{N-1} \left(\frac{\mu}{n^{\alpha_2}} (1-p) + p \right) F_N(T) +$$
$$\tau \tag{4}$$

Where α_2 is as given in section 2. Let $W_{N-n} = \sum_{j=n+1}^{N} X_j$. Then $L_N = L_n + W_{N-n}$. Moreover L_n and W_{N-n} are independent and

$$H_{N-n}(t) = \int_0^{\infty} H_{N-1-n}(\alpha_1(t - y)) dH(y) \tag{5}$$

Where $H_{N-1-n}(t)$ is the distribution of $\sum_{j=n+1}^{N} X_j$ and α_1 is given in section 2 . Since the distribution function of X_{n+1} is $H(t) = F(n^{\alpha_1} t)$, equation (5) can be written, by induction, as $H_{N-n}(t) = F_{N-n}((n+1)^{\alpha_1} t)$. Now $E\left[\chi_{(L_n < T < L_N)} \right] = P(L_n < T < L_n + W_{N-n})$

$$= \int_0^T \int_{T-u}^{\infty} dH_{N-i}(t) dF_n(u)$$

$$= \int_0^T \overline{F}_{N-n}\left((n+1)^{\alpha_1} (T-u) \right) dF_n(u),$$

So that equation (4) becomes

$$E(W) = \left[T\overline{F}_N(T) + \sum_{n=1}^{N-1} \left(\frac{\mu}{n^{\alpha_2}} (1-p) + p \right) \int_0^T \overline{F}_{N-n}\left((n+1)^{\alpha_1}(T-u) \right) dF_n(u) \right.$$

$$\left. + \int_0^T u \, dF_N(u) + \sum_{n=1}^{N-1} \left(\frac{\mu}{n^{\alpha_2}} (1-p) + p \right) F_N(T) + \tau \right]$$

$$= \int_0^T \overline{F}_N(u)\,du + \sum_{n=1}^{N-1}\left(\frac{\mu}{n^{\alpha_2}}(1-p)+p\right)F_N(T)$$

$$+ \sum_{n=1}^{N-1}\left(\frac{\mu}{n^{\alpha_2}}(1-p)+p\right)\int_0^T \overline{F}_{N-n}\big((n+1)^{\alpha_1}(T-u)\big)\,dF_n(u) + \tau.$$

Equation (1) becomes

$$C(T,N)$$
$$= \frac{\left[\begin{array}{c} C\sum_{n=1}^{N-1}\left(\frac{\mu}{n^{\alpha_2}}(1-p)+p\right)\int_0^T \overline{F}_{N-n}\big((n+1)^{\alpha_1}(T-u)\big)\,dF_n(u) - rT\overline{F}_N(T) \\ + \int_0^T C\sum_{n=1}^{N-1}\left(\frac{\mu}{n^{\alpha_2}}(1-p)+p\right)dF_n(u) - r\int_0^T u\,dF_n(u) + R + C_\rho\tau \end{array}\right]}{\left[\begin{array}{c} \int_0^T \overline{F}_N(u)\,du + \sum_{n=1}^{N-1}\left(\frac{\mu}{n^{\alpha_2}}(1-p)+p\right)F_N(T) + \\ \sum_{n=1}^{N-1}\left(\frac{\mu}{n^{\alpha_2}}(1-p)+p\right)\int_0^T \overline{F}_{N-n}\big((n+1)^{\alpha_1}(T-u)\big)\,dF_n(u) + \tau \end{array}\right]}$$

$$= \frac{\left[C\left\{\begin{array}{c}\sum_{n=1}^{N-1}\left(\frac{\mu}{n^{\alpha_2}}(1-p)+p\right)F_N(T) + \sum_{n=1}^{N-1}\left(\frac{\mu}{n^{\alpha_2}}(1-p)+p\right) \\ \times \int_0^T \overline{F}_{N-n}\big((n+1)^{\alpha_1}(T-u)\big)\,dF_n(u)\end{array}\right\} + R + C_\rho\tau - r\int_0^T \overline{F}_N(u)\,du\right]}{\left[\begin{array}{c} \int_0^T \overline{F}_N(u)\,du + \sum_{n=1}^{N-1}\left(\frac{\mu}{n^{\alpha_2}}(1-p)+p\right)F_N(T) + \\ \sum_{n=1}^{N-1}\left(\frac{\mu}{n^{\alpha_2}}(1-p)+p\right)\int_0^T \overline{F}_{N-n}\big((n+1)^{\alpha_1}(T-u)\big)\,dF_n(u) + \tau \end{array}\right]} \quad (6)$$

This is a bivariate function . Obviously, when N is fixed, $C(T,N)$ is a function of T. For fixed $N = m$, it can be written as $C(T,N) = C_m(T), m = 1,2,\ldots\ldots\ldots$. Thus, for a fixed m, we can find T_m^* by analytical methods such that $C_m(T_m^*)$ is minimized. That is when $N = 1,2\ldots m,\ldots\ldots$ we can find $T_1^*, T_2^*, T_3^* \ldots\ldots\ldots T_m^* \ldots\ldots$ respectively, such that the corresponding $C_1(T_1^*), C_2(T_2^*), \ldots\ldots C_m(T_m^*), \ldots\ldots$ are minimized.

Because the total lifetime of the repairable system is limited, the minimum of the long-run average cost per unit time exists. So we can determine the minimum of the long-run average cost per unit time based on $C_1(T_1^*), C_2(T_2^*), \ldots\ldots C_m(T_m^*), \ldots\ldots$For example, the minimum is denoted by $C_n(T_n^*)$. Therefore we may obtain the bivariate optimal replacement policy $(T,N)^*$ such that $C((T,N)^*) = \min_N C_n(T_n^*)$.

Remarks.

(i) When $p = 0$,equation (6) reduces to

$$C(T,N)$$
$$= \frac{\left[C\left\{\sum_{n=1}^{N-1}\left(\frac{\mu}{n^{\alpha_2}}\right)F_N(T) + \sum_{n=1}^{N-1}\left(\frac{\mu}{n^{\alpha_2}}\right)\int_0^T \overline{F}_{N-n}\big((n+1)^{\alpha_1}(T-u)\big)\,dF_n(u)\right\} + R + C_\rho\tau - r\int_0^T \overline{F}_N(u)\,du\right]}{\left[\int_0^T \overline{F}_N(u)\,du + \sum_{n=1}^{N-1}\left(\frac{\mu}{n^{\alpha_2}}\right)F_N(T) + \sum_{n=1}^{N-1}\left(\frac{\mu}{n^{\alpha_2}}\right)\int_0^T \overline{F}_{N-n}\big((n+1)^{\alpha_1}(T-u)\big)\,dF_n(u) + \tau\right]}$$

(ii) When $p = 1$, equation (6) reduces to

$$
C(T, N) = \frac{\left[C(N-1)F_N(T) + \sum_{n=1}^{N-1} \int_0^T \overline{F}_{N-n}\big((n+1)^{\alpha_1}(T-u)\big) dF_n(u) + R + C_\rho \tau - r \int_0^T \overline{F}_N(u) \, du \right]}{\int_0^T \overline{F}_N(u) \, du + (N-1)F_N(T) + \sum_{n=1}^{N-1} \int_0^T \overline{F}_{N-n}\big((n+1)^{\alpha_1}(T-u)\big) dF_n(u) + \tau}
$$

6. CONCLUSION

By considering a bivariate optimal replacement policy for a repairable system with NONN repair times under Alpha series processes, an explicit expression for the long-run average cost per unit time under a univariate replacement policy T is derived. Assuming an alternate repair model, for the same system, expressions for the long-run average cost per unit time under T policy with NONN repair times, N policy with NONN repair times are derived. Furthermore we have derived the long-run average cost per unit time under a bivariate (T, N) policy with NONN repair times. Also, a bivariate optimal replacement policy $(T, N)^*$, such that the long run average cost rate of the system is minimized. The explicit expression of the long-run average cost is derived, the corresponding optimal replacement policy can be determined analytically.

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