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INTRODUCTION TO FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

This paper presents a New Analytical Technique (NAT) for solving nonlinear Fractional Partial Differential Equations (FPDEs) and nonlinear systems of FPDEs. Moreover, the convergence theorem and error analysis of the proposed technique are shown. Analytical and numerical solutions for several examples of nonlinear FPDEs and nonlinear systems of FPDEs are obtained in forms of tables and graphs

Key words: Differential Equations| FPDEs | Fractional Partial Integrals and Derivatives

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HISTORY OF FPDES

The conformable FPDEs are simply PDEs in sense of conformable fractional partial derivatives. Many proposed definitions from a different point of view of applications have been introduced in the literature. Since scientists began to see deficiencies in most of the fractional derivative definitions and why some real-life problems could not be captured and some properties of fractional derivatives are not satisfied (for example, Riemann–Liouville derivative does not satisfy $D^{\alpha}_{u}(1) = 0$ ($D^{\alpha}_{u}(1) = 0$ for the Caputo derivative), if α is not a natural number, all fractional derivatives do not satisfy the known formula of the derivative of the product of two functions: $D\alpha\ a\ (fg) = fD\alpha_{a}(g) + gD\alpha_{a}(f)$, all fractional derivatives do not satisfy the known formula of the derivative of the quotient of two functions

$D_a^{\alpha}(f/g) = \frac{gD_a^{\alpha}(f) - fD_a^{\alpha}(g)}{g^2}$, all fractional derivatives do not satisfy the chain rule:

$D_a^{\alpha}(f \circ g) = f^{(\alpha)}(g(t))g^{(\alpha)}(t)$, all fractional derivatives do not satisfy $D^{\alpha}D^{\beta} f = D^{\alpha+\beta} f$ in general and the Caputo definition assumes that the function f is differentiable). Consequently, a rather simple and intriguing definition called "the conformable fractional derivative" was proposed by Khalil et al. in 2014 (see, 1]) which is depending just on the basic limit definition of the derivative. Namely, for a function $f : (0,\infty) \to R$, the (conformable) fractional derivative of order $0 < \alpha \leq 1$ of f at $t > 0$ is and the fractional derivative at $t = 0$ is defined as ('r$_\alpha$ f)(0) = limε→0+('rα f)(t).

Through this a new definition, many most recent works have been done, see for example, recently, Yücel Çenesiz et al. (see, [2]) studied the nonlinear partial differential equations with conformable derivative using the first integral method, K. Hosseini et al. (see, [3]) applied the modified Kudryashov method for solving the conformable time fractional Klein–Gordon equations, Hadi Rezazadeh et al. (see, [4]) obtained traveling wave solution of conformable fractional generalized reaction duffing model by generalized projective Riccati equation method, Farid Samsami Khodadad et al. (see, [5]) proposed the Riccati sub equation method to obtain the soliton solution of the conformable fractional Zakharov–Kuznetsov equation.

Since the solutions to a large class of FPDEs are rarely available, approximate and numerical methods are applicable. Therefore, accurate methods for finding the solutions of FPDEs are yet under investigation. Some analytical and numerical methods for solving FPDEs exist in the literature for example, the modified extended tanh-function method (see, [6]) where the method employed to solve fractional partial differential equations by turning them into nonlinear ordinary differential equations of integer orders, the HPM (see, [7]) where the authors applied this method for solving nonlinear partial differential equations with fractional time derivative, homotopy perturbation technique (see, [8]) where the idea of this technique was to utilize both the initial and boundary conditions in the recursive relation for obtaining approximate solution, variational iteration method and decomposition method (see, [9]) where these two methods applied to obtain the approximate solution of nonlinear fractional order partial differential equations and so on.

$$\mathcal{T}_\alpha f = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}, \tag{1.1.7}$$

Recently, T. Bakkyaraj and R. Sahadevan (see,10) applied homotopy analysis method to obtain the approximate analytical solution of two coupled time fractional nonlinear Schrödinger equations. Zigen Ouyang (see,11) obtained some conditions for the existence of the solution of a class of nonlinear fractional order partial differential equations with delay. Rihuan et al. (see,12) proposed the fast direct method for solving the linear block lower triangular Toeplitz-like with tridiagonal blocks system which arises from the time fractional partial differential equation.

Fractional Partial Integrals and Derivatives

There are various definitions and theorems of fractional integrals and derivatives. In this paper, we give some definitions and theorems of the fractional calculus theory.

Consider a locally integrable[1] real-valued function $u : G \times R \to R$ whose domain of definition $G \subset R$ is an region with $-\infty \leq a < t < b \leq \infty$. Then

$${}_a\mathfrak{I}_t u(x,t) = \int_a^t u(x,\tau)d\tau. \tag{1.2.1}$$

By integrating (1.2.1) $(n - 1)$-times, we get the following fundamental formula

$$
\begin{aligned}
{}_a\mathfrak{J}_t^n u(x,t) &= \int_a^t \int_a^{t_1} \cdots \int_a^{t_{n-1}} u(x,t_n)dt_n dt_{n-1} \cdots dt_2 dt_1 \\
&= \frac{1}{(n-1)!} \int_a^t (t-\tau)^{n-1} u(x,\tau) d\tau, \qquad (1.2.2)
\end{aligned}
$$

$$
\begin{aligned}
{}_a\mathfrak{J}_t^n u(x,t) &= \int_a^t \int_a^{t_1} \cdots \int_a^{t_{n-1}} u(x,t_n)dt_n dt_{n-1} \cdots dt_2 dt_1 \\
&= \frac{1}{(n-1)!} \int_a^t (t-\tau)^{n-1} u(x,\tau) d\tau, \qquad (1.2.2)
\end{aligned}
$$

where $\alpha < t < \beta$ and $n \in \mathbb{N}$. By replacing n in (1.2.2) with noninteger positive q, we reach 1A function $u : G \times R \rightarrow R$ is called locally integrable if it is integrable on all compact subsets $K \subset G$ to the following Riemann–Liouville definition for fractional integral.

$$
{}^{RL}\mathfrak{D}_t^q u(x,t) = \frac{1}{\Gamma(m-q)} \frac{\partial^m}{\partial t^m} \int_0^t (t-\tau)^{m-q-1} u(x,\tau) d\tau, \; t > 0, \qquad (1.2.4)
$$

Definition 1.2.1. Let q,t € R, m _ 1 ≤ q < m € N, then the Riemann–Liouville time fractional partial integral of order qfor a function u(x,t) is defined by

$$
\begin{cases}
\mathfrak{J}_t^q u(x,t) = \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-\tau)^{q-1} u(x,\tau) d\tau, \; t > 0, \\
\mathfrak{J}_t^0 u(x,t) = u(x,t), \quad q = 0, \; t > 0,
\end{cases} \qquad (1.2.3)
$$

where Γ is the well-known Gamma function.

Definition 1.2.2. Let q,t € R, m _ 1 ≤ q < m € N, then the Riemann-Liouville time fractional partial derivative of order q for a function u(x,t) is defined by

where Γ is the well-known gamma function.

Definition 1.2.3. Let q,t € R and m _ 1 < q < m € N, then the Caputo time fractional partial derivative of order q for a function u(x,t) is defined by

$$
\begin{cases}
\mathfrak{D}_t^q u(x,t) = \dfrac{1}{\Gamma(m-q)} \displaystyle\int_0^t (t-\tau)^{m-q-1} \frac{\partial^m u(x,\tau)}{\partial \tau^m} d\tau, \; t > 0, \\
\mathfrak{D}_t^q u(x,t) = \dfrac{\partial^m u(x,t)}{\partial t^m}, q = m \in \mathbb{N}.
\end{cases} \qquad (1.2.5)
$$

Definition 1.2.4. Fort = a+mh,h =~ 0,t > a and m _ 1 < q < m E N. The Grunwald-Letnikov fractional partial derivative for a function u(x,t) is defined by

$$ {}^{GL}\mathcal{D}_t^q u(x,t) = \lim_{h\to 0} h^{-q} \sum_{j=0}^{\frac{t-a}{h}} (-1)^j \binom{q}{j} u(x,t-jh). \tag{1.2.6}$$

Theorem 1.2.1. *Let q,t € R, m−1 < q < m € N and t > 0. Then*

$$\begin{cases} \mathcal{J}_t^q \mathcal{D}_t^q u(x,t) = u(x,t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} \frac{\partial^k u(x,0^+)}{\partial t^k}, \\ \mathcal{D}_t^q \mathcal{J}_t^q u(x,t) = u(x,t). \end{cases} \tag{1.2.7}$$

Theorem 1.2.2. *Let q,t € R, m−1 < q < m € N and t > 0. Then*

$$\begin{cases} \mathcal{D}_t^q t^p = \frac{\Gamma(p+1)}{\Gamma(p-q+1)} t^{p-q}, \; m \le p, \; p \in \mathbb{R}, \\ \mathcal{D}_t^q t^p = 0, \; p \le m-1. \end{cases} \tag{1.2.8}$$

Theorem 1.2.3. Let q1,q2 € R, such that n − 1 < q1 < n, m − 1 < q2 < m, n =~ m for n,m € N. Then, in general

$$\begin{cases} \mathcal{D}_t^{q_1} \mathcal{D}_t^{q_2} u(x,t) \neq \mathcal{D}_t^{q_2} \mathcal{D}_t^{q_1} u(x,t) \neq \mathcal{D}_t^{q_1+q_2} u(x,t), \\ \mathcal{D}_t^{q_1} \mathcal{D}_t^{m} u(x,t) \neq \mathcal{D}_t^{m} \mathcal{D}_t^{q_1} u(x,t). \end{cases} \tag{1.2.9}$$

Theorem 1.2.4. Assume that a function u(x,t) is infinitely ɑ-differentiable function, for some 0 < ɑ ≤ 1 at a neighborhood of a point (x,t0),t0 > 0. Then u(x,t) has the fractional power series expansion:

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} \left[\mathcal{D}_t^{k\alpha} u(x,t)\right]_{(x,t_0)} (t-t_0)^{k\alpha}, \tag{1.2.10}$$

where $D_t^{k\alpha}$ denotes the application of time fractional partial derivative for k-t

Definition 1.2.5. Let a function u(x,t) be infinitely ɑ-differentiable function. Then, the time FPDTfor u(x,t) is defined by

$${}_t\bar{U}_\alpha(x,k) = \frac{1}{\Gamma(k\alpha+1)} \left[\mathcal{D}_t^{k\alpha} u(x,t)\right]_{(x,t_0)}, t_0 > 0, \tag{1.2.11}$$

For some α€ (0,1)

Definition 1.2.6. Let a function u(x,t) be infinitely β-differentiable function. Then, the space FPDTfor u(x,t) is defined by

for some β ∈ (0,1] where$D^{h\beta}_x$ denotes the application of space fractional partial derivative

for h-times.

Theorem 1.2.5. Assume that a function u(x,t) is an infinitely α,β-differentiable function, for some α,β ∈ R,α,β > 0. Then u(x,t) has the following fractional power series expansion:

$$ {}_x\bar{U}_\beta(x,k) = \frac{1}{\Gamma(h\beta+1)}\left[\mathcal{D}_x^{h\beta}u(x,t)\right]_{(x_0,t)}, x_0 > 0, \tag{1.2.12}$$

where Dkαx and Dhβt denote the applications of space and time fractional partial derivatives respectively for k and h-times.

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{\left[\mathcal{D}_x^{k\alpha}\mathcal{D}_t^{h\beta}u(x,t)\right]_{(x_0,t_0)}}{\Gamma(k\alpha+1)\Gamma(h\beta+1)}(x-x_0)^{k\alpha}(t-t_0)^{h\beta},\ x_0,t_0>0, \tag{1.2.13}$$

Definition 1.2.7. Let a function u(x,t) be an infinitely α,β-differentiable function. Then, the generalized FPDT for u(x,t) is defined by

$$\bar{U}_{\alpha,\beta}(k,h) = \frac{\left[\mathcal{D}_x^{k\alpha}\mathcal{D}_t^{h\beta}u(x,t)\right]_{(x_0,t_0)}}{\Gamma(k\alpha+1)\Gamma(h\beta+1)}, x_0,t_0>0, \tag{1.2.14}$$

for some α,β ∈ R,α,β > 0.

2. New Analytical Technique for Solving Nonlinear Fractional Partial Differential Equations

In this section, we present a NAT to solve nonlinear FPDEs with initial values of the following form

$$\begin{cases} \mathcal{D}_t^q u(\bar{x},t) = f(\bar{x},t) + L(u(\bar{x},t)) + N(u(\bar{x},t)),\ m-1<q<m\in\mathbb{N}, \\ \dfrac{\partial^r u(\bar{x},0)}{\partial t^r} = f_r(\bar{x}),\ \ r=0,1,2,\ldots,m-1, \end{cases} \tag{2.2.1}$$

where $L(u(\bar{x},t))$ and $N(u(\bar{x},t))$ are linear and nonlinear operators, respectively, of $u(\bar{x},t)$ and its partial derivatives which might include other fractional partial derivatives, and $f(\bar{x},t)$, $f_r(\bar{x})$ are known analytic functions, D^q_t is the Caputo time fractional partial derivative of order q, and $\bar{x} = (x_1,x_2,\ldots,x_n) \in R^n$.

The NAT has a computational power in obtaining piecewise analytical solutions for nonlinear FPDEs . To introduce the ANT, first we need to present the following results.

Theorem 2.2.1. ***Let*** $u(\bar{x},t) = \sum_{k=0}^{\infty} u_k(\bar{x},t)$, *we define* $u_\lambda(\bar{x},t) = \sum_{k=0}^{\infty}\lambda^k u_k(\bar{x},t)$ *where* $\lambda \in$

Whereλ∈[0,1]isarameter.

$$L(u_\lambda(\bar{x},t)) = L(\sum_{k=0}^{\infty} \lambda^k u_k(\bar{x},t)) = \sum_{k=0}^{\infty} \lambda^k L(u_k(\bar{x},t)). \tag{2.2.2}$$

Then, the linear operator L(uλ (̄x,t)) satisfies the following property: [0,1] is a parameter.

Theorem 2.2.2.

Let $(\bar{x},t) = \sum_{k=0}^{\infty} u_k(\bar{x},t)$, *we define* $u_\lambda(\bar{x},t) = \sum_{k=0}^{\infty} \lambda^k u_k(\bar{x},t)$ *where* $\lambda \in$ u

[0; 1] is a parameter. Then, the nonlinear operator N(ul (x ̄;t)) satisfies the following property

$$N(u_\lambda(\bar{x},t)) = N(\sum_{k=0}^{\infty} \lambda^k u_k(\bar{x},t)) = \sum_{n=0}^{\infty} \left[\frac{1}{n!}\frac{\partial^n}{\partial \lambda^n}[N(\sum_{k=0}^{n} \lambda^k u_k(\bar{x},t))]_{\lambda=0}\right]\lambda^n. \tag{2.2.3}$$

Proof. By using Maclaurin expansion for $N(\sum_{k=0}^{\infty} \lambda^k u_k(\bar{x},t))$ with respect to λ, we have

$$\begin{aligned}
N(u_\lambda(\bar{x},t)) &= [N(\sum_{k=0}^{\infty} \lambda^k u_k(\bar{x},t))]_{\lambda=0} + [\frac{\partial}{\partial \lambda}[N(\sum_{k=0}^{\infty} \lambda^k u_k(\bar{x},t))]_{\lambda=0}]\lambda \\
&\quad + [\frac{1}{2!}\frac{\partial^2}{\partial \lambda^2}[N(\sum_{k=0}^{\infty} \lambda^k u_k(\bar{x},t))]_{\lambda=0}]\lambda^2 + \cdots \\
&= \sum_{n=0}^{\infty} [\frac{1}{n!}\frac{\partial^n}{\partial \lambda^n}[N(\sum_{k=0}^{\infty} \lambda^k u_k(\bar{x},t))]_{\lambda=0}]\lambda^n \\
&= \sum_{n=0}^{\infty} \left[\frac{1}{n!}\frac{\partial^n}{\partial \lambda^n}[N(\sum_{k=0}^{n} \lambda^k u_k(\bar{x},t) + \sum_{k=n+1}^{\infty} \lambda^k u_k(\bar{x},t))]_{\lambda=0}\right]\lambda^n \\
&= \sum_{n=0}^{\infty} \left[\frac{1}{n!}\frac{\partial^n}{\partial \lambda^n}[N(\sum_{k=0}^{n} \lambda^k u_k(\bar{x},t) + \lambda^{n+1}u_{n+1}(\bar{x},t) + \lambda^{n+2}u_{n+2}(\bar{x},t) + \cdots)]_{\lambda=0}\right]\lambda^n \\
&= \sum_{n=0}^{\infty} \left[\frac{1}{n!}\frac{\partial^n}{\partial \lambda^n}[N(\sum_{k=0}^{n} \lambda^k u_k(\bar{x},t))]_{\lambda=0}\right]\lambda^n.
\end{aligned}$$

Since $\frac{\partial^n}{\partial \lambda^n}\left[N(\lambda^{n+1}u_{n+1}(\bar{x},t) + \lambda^{n+2}u_{n+2}(\bar{x},t) + \cdots)\right]_{\lambda=0} = 0.$ □

Definition 2.2.1. The polynomial E_n(u0(x ̄;t); u1(x ̄;t); : : : ; u_n(x ̄;t)) can be defined by

$$E_n(u_0(\bar{x},t), u_1(\bar{x},t), \ldots, u_n(\bar{x},t)) = \frac{1}{n!}\frac{\partial^n}{\partial \lambda^n}\left[N(\sum_{k=0}^{n} \lambda^k u_k(\bar{x},t))\right]_{\lambda=0}. \tag{2.2.4}$$

:

Remark 2.1. Let E_n = E_n(u0(x ̄;t); u1(x ̄;t); : : : ; u_n(x ̄;t)) be as in Definition 2.2.1. Then, by using Theorem 2.2.2, the nonlinear operator $N(u\lambda(\bar{x},t))$ can be expressed in terms of E_n as:

$$N(u_\lambda(\bar{x},t)) = \sum_{n=0}^{\infty} \lambda^n E_n. \tag{2.2.5}$$

The following theorem introduces an analytical solution obtain by a NAT for the nonlinear fractional partial differential equation given by (2.2.1).

Theorem 2.2.3 (Existence Theorem). Let $m-1 < q < m \in \mathbb{N}$ and $f(\bar{x},t)$, $f_k(\bar{x})$ be known analytic functions. Then, the equation (2.2.1) admits a solution given by

$$u(\bar{x},t) = f_t^{(-q)}(\bar{x},t) + \sum_{r=0}^{m-1} \frac{t^r}{r!} f_r(\bar{x}) + \sum_{k=1}^{\infty} [L_t^{(-q)}(u_{(k-1)}) + E_{(k-1)_t}^{(-q)}(u_0,u_1,\ldots,u_{k-1})], \tag{2.2.6}$$

where $f_t^{(-q)}(\bar{x},t), L_t^{(-q)}(u_{(k-1)})$ and $E_{(k-1)_t}^{(-q)}(u_0,u_1,\ldots,u_{k-1})$ denote the time fractional partial integral of order q for $f(\bar{x},t)$, $L(u_{(k-1)})$ and $E_{(k-1)}(u_0,u_1,\ldots,u_{k-1})$ respectively.

Proof. We assume that a solution function $u(\bar{x},t)$ of equation (2.2.1) has the following analytic expansion:

$$u(\bar{x},t) = \sum_{k=0}^{\infty} u_k(\bar{x},t). \tag{2.2.7}$$

To solve the nonlinear fractional partial differential equation (2.2.1), we consider

$$\mathcal{D}_t^q u_\lambda(\bar{x},t) = \lambda[f(\bar{x},t) + L(u_\lambda(\bar{x},t)) + N(u_\lambda(\bar{x},t))],\ \lambda \in [0,1], \tag{2.2.8}$$

with the initial condition given by

$$\frac{\partial^r u_\lambda(\bar{x},0)}{\partial t^r} = g_r(\bar{x}),\quad r = 0,1,2,\ldots,m-1. \tag{2.2.9}$$

By taking Riemann-Liouville time fractional partial integral of order q to both sides of the solution given by

$$u_\lambda(\bar{x},t) = \sum_{k=0}^{\infty} \lambda^k u_k(\bar{x},t). \tag{2.2.10}$$

Next, we assume that the nonlinear fractional partial differential equation (2.2.8) has a initial value problem (2.2.8) and by using Theorem 1.2.1,
we obtain

$$u_\lambda(\bar{x},t) = \quad r \quad \partial^r u_\lambda(\bar{x},0) + \lambda \, \mathfrak{I}^q_t [f(\bar{x},t) + L(u_\lambda) + N(u_\lambda)]. \tag{2.2.11}$$

By using the initial condition given by (2.2.1), the equation (2.2.11) can be rewritten as:

$$u_\lambda(\bar{x},t) = \sum_{r=0}^{m-1} \frac{t^r}{r!} g_r(\bar{x}) + \lambda \left[f_t^{(-q)}(\bar{x},t) + \mathfrak{I}_t^q [L(u_\lambda(\bar{x},t))] + \mathfrak{I}_t^q [N(u_\lambda(\bar{x},t))] \right]. \tag{2.2.12}$$

Further, by substituting (2.2.10) into (2.2.12), we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} \lambda^k u_k(\bar{x},t) = {} & \sum_{r=0}^{m-1} \frac{t^r}{r!} g_r(\bar{x}) + \lambda \left[f_t^{(-q)}(\bar{x},t) \right. \\ & \left. + \mathfrak{I}_t^q [L(\sum_{k=0}^{\infty} \lambda^k u_k(\bar{x},t))] + \mathfrak{I}_t^q [N(\sum_{k=0}^{\infty} \lambda^k u_k(\bar{x},t))] \right]. \end{aligned} \tag{2.2.13}$$

By using Theorem 2.2.1 and Theorem 2.2.2, the equation (2.2.13) becomes

$$\begin{aligned} \sum_{k=0}^{\infty} \lambda^k u_k(\bar{x},t) = {} & \sum_{r=0}^{m-1} \frac{t^r}{r!} g_r(\bar{x}) + \lambda f_t^{(-q)} + \mathfrak{I}_t^q \lambda \sum_{k=0}^{\infty} \lambda^k [L(u_k(\bar{x},t))] \\ & + \mathfrak{I}_t^q \lambda \sum_{n=0}^{\infty} \left[\frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} [N(\sum_{k=0}^{n} \lambda^k u_k(\bar{x},t))]_{\lambda=0} \right] \lambda^n. \end{aligned} \tag{2.2.14}$$

Next, we use Definition 2.2.1 and Remark 2.1 in the equation (2.2.14), we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} \lambda^k u_k(\bar{x},t) = {} & \sum_{r=0}^{m-1} \frac{t^r}{r!} g_r(\bar{x}) + \lambda f_t^{(-q)}(\bar{x},t) \\ & + \mathfrak{I}_t^q \lambda \sum_{k=0}^{\infty} [\lambda^k L(u_k(\bar{x},t))] + \mathfrak{I}_t^q \lambda \sum_{n=0}^{\infty} E_n \lambda^n. \end{aligned} \tag{2.2.15}$$

By equating the terms in the equation (2.2.15) with identical powers of λ, we obtain

$$\begin{cases} u_0(\bar{x},t) = \sum_{r=0}^{m-1} \frac{t^r}{r!} g_r(\bar{x}), \; u_1(\bar{x},t) = f_t^{(-q)}(\bar{x},t) + L_t^{(-q)} u_0(\bar{x},t) + E_{0t}^{(-q)}, \\ u_k(\bar{x},t) = L_t^{(-q)} u_{(k-1)}(\bar{x},t) + E_{(k-1)t}^{(-q)}, \; k = 2,3,\ldots. \end{cases} \tag{2.2.16}$$

Next, by substituting (2.2.16) in the equation (2.2.10), we obtain the solution of the equation (2.2.8). Now, from the equations (2.2.7) and (2.2.10), we obtain

$$u(\bar{x},t) = \lim_{\lambda \to 1} u_\lambda(\bar{x},t) = u_0(\bar{x},t) + u_1(\bar{x},t) + \sum_{k=2}^{\infty} u_k(\bar{x},t). \tag{2.2.17}$$

From (2.2.17) and by using the initial conditions, we see that

$$\frac{\partial^k u(\bar{x},0)}{\partial t^k} = \lim_{\lambda\to 1} \frac{\partial^k u_\lambda(\bar{x},0)}{\partial t^k}. \tag{2.2.18}$$

Which implies that g(x¯) = f (x¯). By using (2.2.16) into (2.2.17), we complete the proof

In the following theorems, we establish the convergence and maximum absolute error results of the analytical solution given by (2.2.6) for the nonlinear fractional partial differential equation given by (2.2.1).

Theorem 2.2.4 (Convergence Theorem). Let B be a Banach space. Then, the series solution of the equation (2.2.16) converges to S ∈ B if there exists γ, 0 ≤ γ < 1 such that, ~unM≤ γMu(n−1)M for ∀n ∈ N.

Proof. We define the sequence S_n of the following partial sums:

$$\begin{cases} S_0 = u_0(\bar{x},t),\ S_1 = u_0(\bar{x},t)+u_1(\bar{x},t), \\ S_2 = u_0(\bar{x},t)+u_1(\bar{x},t)+u_2(\bar{x},t), \\ \vdots \\ S_n = u_0(\bar{x},t)+u_1(\bar{x},t)+u_2(\bar{x},t)+\cdots+u_n(\bar{x},t). \end{cases} \tag{2.2.19}$$

We need to show that $\{S_n\}$ is a Cauchy sequence in a Banach space B. For this purpose, we consider

$$\|S_{n+1}-S_n\| = \|u_{n+1}(\bar{x},t)\| \leq \gamma\|u_n(\bar{x},t)\| \leq \gamma^2\|u_{n-1}(\bar{x},t)\| \leq \cdots \gamma^{n+1}\|u_0(\bar{x},t)\|. \tag{2.2.20}$$

For every $n,n' \in \mathbb{N}$, $n \geq n'$, by using the equation (2.2.20) and triangle inequality successively, we have

$$\begin{aligned} \|S_n - S_{n'}\| &= \|S_{n'+1}-S_{n'}+S_{n'+2}-S_{n'+1}+\cdots+S_n-S_{n-1}\| \\ &\leq \|S_{n'+1}-S_{in'}\|+\|S_{n'+2}-S_{n'+1}\|+\cdots+\|S_n-S_{n-1}\| \\ &\leq \gamma^{n'+1}\|u_0(\bar{x},t)\|+\gamma^{n'+2}\|u_0(\bar{x},t)\|+\cdots+\gamma^{n}\|u_0(\bar{x},t)\| \\ &= \gamma^{n'+1}(1+\gamma+\cdots+\gamma^{n-n'-1})\|u_0(\bar{x},t)\| \\ &\leq \gamma^{n'+1}\left(\frac{1-\gamma^{n-n'}}{1-\gamma}\right)\|u_0(\bar{x},t)\|. \end{aligned} \tag{2.2.21}$$

1 − γ

Since $0 < \gamma < 1$, so $1 -¥^{n-n'} \leq 1$. Then

$$\|S_n - S_{n'}\| \leq \frac{\gamma^{n'+1}}{1-\gamma}\|u_0(\bar{x},t)\|. \qquad (2.2.22)$$

Since $u0(\bar{}x,t)$ is bounded, then

$$\lim_{n,n'\to\infty} \|S_n - S_{n'}\| = 0. \qquad (2.2.23)$$

Therefore, the sequence $\{S_n\}$ is a Cauchy sequence in a Banach space B and the series solution defined in the equation (2.2.17) converges. This completes the proof

Theorem 2.2.5 (Error Analysis). The maximum absolute truncation error for the series of solution (2.2.7) for nonlinear FPDEs (2.2.1) is estimated to be

$$\sup_{(\bar{x},t)\in\Omega} \left|u(\bar{x},t) - \sum_{k=0}^{n'} u(\bar{x},t)\right| \leq \frac{\gamma^{n'+1}}{1-\gamma} \sup_{(\bar{x},t)\in\Omega} |u_0(\bar{x},t)|, \qquad (2.2.24)$$

where the region $\Omega \subset R$ n+1

Proof. From Theorem 2.2.4, we have

$$\|S_n - S_{n'}\| \leq \frac{\gamma^{n'+1}}{1-\gamma} \sup_{(\bar{x},t)\in\Omega} |u_0(\bar{x},t)|. \qquad (2.2.25)$$

But we assume that $Sn = \sum n\ k=0\ u(\bar{}x,t)$ and since $n \to \infty$, we obtain $S_n \to u(\bar{}x,t)$, then the equation (2.2.25) can be rewritten as:

$$\|u(\bar{x},t) - S_{n'}\| = \left\|u(\bar{x},t) - \sum_{k=0}^{n'} u(\bar{x},t)\right\| \leq \frac{\gamma^{n'+1}}{1-\gamma} \sup_{(\bar{x},t)\in\Omega} |u_0(\bar{x},t)|. \qquad (2.2.26)$$

Therefore, the maximum absolute error is

$$\sup_{(\bar{x},t)\in\Omega} \left|u(\bar{x},t) - \sum_{k=0}^{n'} u(\bar{x},t)\right| \leq \frac{\gamma^{n'+1}}{1-\gamma} \sup_{(\bar{x},t)\in\Omega} |u_0(\bar{x},t)|, \qquad (2.2.27)$$

And this completes the proof

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