Malaya Journal of Matematik, Vol. S, No. 1, 419-421, 2021
https://doi.org/10.26637/MJMS2101/0094

Non - trivial integer solutions of the quadratic diophantine equations $x^2 = 25y^2 + 29z^2$ and $\mu^2 + \vartheta^2 = 61\phi^2$

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Abstract
Many types of research have been devoted to finding the solutions η, ζ, δ in the set of non-negative integers of Diophantine equations of the types $\eta^2 = \alpha\zeta^2 + \beta\delta^2$ and $\eta^2 + \zeta^2 = \gamma\delta^2$, where the fixed values $\eta, \zeta, \gamma, \alpha, \beta$ and δ are integers. In this article, we will discuss integral solutions of the ternary quadratic Diophantine equation representing infinite cone given by $x^2 = 25y^2 + 29z^2$ and lattice points of the ternary homogeneous Diophantine equation representing an infinite cone given by $\mu^2 + \vartheta^2 = 61\phi^2$ are analyzed for its non-zero distinct lattice points. Few different patterns of integer points satisfying the infinite cone under consideration are obtained.

Keywords
Diophantine Equation, Infinite Cone, Integral Solutions, Homogenous Equation, Non- zero Integers.

AMS Subject Classification
11D09, 11D72.

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Article History: Received **01** November **2020**; Accepted **10** January **2021**

Contents

1. Introduction

Number theory is the branch of Mathematics concerned with studying the properties and relations of integers. Many of these problems are concerned with the properties of prime numbers. Number theory includes the different aspects of natural numbers and their extensions in various fields of Mathematics and Science. There are number of branches of number theory including algebraic number theory, analytic number theory, geometric number theory, combinatorial number theory, computational number theory, probabilistic number theory and so on Number theory also includes the study of irrational numbers, transcendental numbers, continued fractions and Diophantine equation [1-5].

2. Integral Solutions of an Infinite Elliptic Cone $x^2 = 25y^2 + 29z^2$

In this section, the ternary quadratic Diophantine equation $x^2 = 25y^2 + 29z^2$ is analyzed for various patterns of integer solutions. Pictorial representation of the equation

$$\left(x^2 = 25y^2 + 29z^2\right).$$

2.1 Method of Analysis

The quadratic Diophantine equation with three unknowns to be solved is

$$x^2 = 25y^2 + 29z^2 \tag{2.1}$$

We present below different patterns of integral solutions of (2.1):
Pattern 1:
Consider (2.1) as

$$x^2 = 29z^2 + (5y)^2 \tag{2.2}$$

The solutions are found to be

$$x = 29m^2 + n^2$$
$$5y = 29m^2 - n^2$$
$$z = 2mn, \quad \text{where } m, n \in N.$$

Since our interest centers on finding integral solutions, we consider the following case:
Case (i): Both m and n are even.
(i.e.), take $m = 2p, n = 2q, \quad p, q \in Z$. Hence, the solutions of (2.1) are given by

$$x = 29(2p)^2 + (2q)^2$$
$$5y = 29(2p)^2 - (2q)^2$$
$$Z = 8pq$$

In general,

$$x = 2^n\left(29p^2 + q^2\right)$$
$$5y = 2^n\left(29p^2 - q^2\right)$$
$$z = 2^{n+1}(pq), \quad n \in N$$

Pattern 2:
Consider (2.1) as:

$$x^2 - 25y^2 = 29z^2 \tag{2.3}$$

which can be written in the form of ratio as

$$\frac{x - 5y}{29z} = \frac{z}{x + 5y} = \frac{p}{q} \text{ (say) }, q \neq 0$$

Expressing this as a system of simultaneous equations

$$qx - 5qy - 29pz = 0 \tag{2.4}$$
$$px + 5py - qz = 0 \tag{2.5}$$

and solving (2.4) and (2.5) we get

$$x = 145p^2 + 5q^2$$
$$y = -29p^2 + q^2$$
$$z = 10pq, \text{ where } p, q \in N$$

Pattern 3:
(2.3) can also be written as:

$$\frac{x + 5y}{z} = \frac{29z}{x - 5y} = \frac{p}{q}(\text{ say }), q \neq 0$$

Proceeding as above, we obtain

$$x = -5p^2 - 145q^2$$
$$y = -p^2 + 29q^2$$
$$z = -10pq, \text{ where } p, q \in N$$

Pattern 4: Consider (2.1) as

$$x^2 - 29z^2 = 25y^2 \tag{2.6}$$

Assume y as

$$y = a^2 - 29b^2 \tag{2.7}$$

write 25 as

$$25 = p^2 - 29q^2 \tag{2.8}$$

in the above equation

$$\left(x^2 - 29z^2\right) = 25\left(a^2 - 29b^2\right)^2 \tag{2.9}$$

Because of (2.7) and (2.8),(2.9) is factorizable form as

$$(x + \sqrt{29}z)(x - \sqrt{29}z) = 25(a + \sqrt{29}b)^2(a - \sqrt{29}b)^2$$

Define, $x + \sqrt{29}z = (p + \sqrt{29}q)(a + \sqrt{29}b)^2$. Equating the rational and irrational parts, we get

$$x = p\left(a^2 + 29b^2\right) + 58abq, z = q\left(a^2 + 29b^2\right) + 2abp \tag{2.10}$$

Thus, (2.7) and (2.10) represent the general solution of (2.1).

To find the value of p, q we proceed as follows: Let (p_0, q_0) be the initial solution of $p^2 - 29q^2 = 25$ Employing the linear transformation $p = p_0 + 170h, q = q_0 + 38h$ repeatedly, the other values of p and q satisfying the equation (2.8) are given by $p = -57799p_0 + 122740q_0$, $q = -12920p_0 + 27437q_0$.
Numerical example is presented below:

	An algebraic expression for 25	Solutions
Case I	$(237 - 44\sqrt{29})$ $(237 + 44\sqrt{29})$	$x = 237\left(a^2 + 29b^2\right) + 2552ab$ $y = a^2 - 29b^2$ $z = 44\left(a^2 + 29b^2\right) + 474ab$

In the above section, we have presented four different patterns of non-zero distinct integer solutions of the infinite elliptic cone given by $x^2 = 25y^2 + 29z^2$. One may search for other patterns of non-zero distinct integer solutions for the same equation.
Pattern 4:
Instead of (3.8) we can also write 1 as

$$1 = \frac{(\sqrt{61} + 5)(\sqrt{61} - 5)}{36}$$

Proceeding as above, we obtain the solutions as

$$\mu = \mu(A, B) = 36\left(61A^2 - B^2\right)$$
$$\vartheta = \vartheta(A, B) = 6\left[5\left(61A^2 + B^2\right) + 122AB\right]$$
$$\phi = \phi(A, B) = 6\left[61A^2 + B^2 + 10AB\right]$$

Pattern 6:
Equation (2.1) can be written as

$$\frac{\mu + 5\phi}{6\phi - \vartheta} = \frac{6\phi + \vartheta}{\mu - 5\phi} = \frac{p}{q}, (say), q \neq 0$$

Proceeding as above, we obtain the solutions as

$$\mu = \mu(p,q) = -5p^2 + 5q^2 - 12pq$$
$$\vartheta = \vartheta(p,q) = -6p^2 + 6q^2 + 10pq$$
$$\phi = \phi(p,q) = -p^2 - q^2, p, q \in N$$

Pattern 7:

Let $\mu = 5u + 6v$ and $\vartheta = 6u - 5v$. Then $\mu^2 + \vartheta^2 = (61)\phi^2$ and equation (2.1) becomes $u^2 + v^2 = \phi^2$. We obtain $u = k\left(m^2 - n^2\right), v = 2kmn, \phi = k\left(m^2 + n^2\right)$, for some integers k, m and n, hence the solutions

$$\mu = k\left(5m^2 + 12mn - 5n^2\right)$$
$$\vartheta = k\left(6m^2 - 10mn - 6n^2\right)$$
$$\phi = k\left(m^2 + n^2\right), \text{ where } \quad m, n \in N.$$

Pattern 8:

Let $\mu = 5u - 6v$ and $\vartheta = 6u + 5v$. Then $\mu^2 + \vartheta^2 = 61\left(u^2 + v^2\right)$ and equation (1) becomes $u^2 + v^2 = \phi^2$.

We obtain $u = k\left(m^2 - n^2\right), v = 2kmn, z = k\left(m^2 + n^2\right)$, for some integers k, m and n, hence the solutions

$$\mu = k\left(5m^2 - 12mn - 5n^2\right)$$
$$\vartheta = k\left(6m^2 + 10mn - 6n^2\right)$$
$$\phi = k\left(m^2 + n^2\right), \text{ where } m, n \in N.$$

In this above section, we have made an attempt to obtain a complete set of non-trivial distinct lattice points of an infinite cone.

3. Conclusion

To conclude, one may search for other choices of solutions to the considered infinite cone and further Diophantine equations with multi variables. Since the ternary quadratic Diophantine equations are rich in variety, one may search for other choices of Diophantine equations to find their corresponding integer solutions.

Acknowledgement

We would like to show our gratitude to Dr. Manju Somanath, Assistant Professor of Mathematics, National College and Prof. M.A.Gopalan, Professor of Mathematics, Shrimati Indira Gandhi College, for sharing their pearls of wisdom.

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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