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Article *in* Archives of Computational Methods in Engineering · July 2022

DOI: 10.1007/s11831-022-09779-8

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Archives of Computational Methods in Engineering
https://doi.org/10.1007/s11831-022-09779-8

REVIEW ARTICLE

A Systematic Review on Generalized Fuzzy Numbers and Its Applications: Past, Present and Future

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Received: 14 December 2021 / Accepted: 7 May 2022

Abstract
Nowadays, the program's mechanisms are becoming more dynamic. As a result, maintaining performance for a longer duration of time to increase the system's long-term growth is tough. This is mostly because of the malfunction occurrence during the study, as the machine does not always have all details. In order to tackle this problem, the available data must be used to construct the problems. However, one of the most successful data theories is fuzzy set theory. The concept of fuzzy logic has recently grown in favour, and it plays an important role in engineering and management. Fuzzy arithmetic was very significant in research fields including decision-making problems, confidence analysis, optimization etc. as compare to others. Fuzzy numbers came into existence to perform operations on fuzzy observations. The distinction between generalised fuzzy and classic fuzzy arithmetic operations is that the former can handle both non-normal and normalised fuzzy, while the latter can only handle normalised fuzzy.
The goal of this research is to give a broad overview of current techniques in this field. The methodology reported in this article focuses on improving the arithmetic process in a fuzzy environment. The current arithmetic operations take into account the same degree of precision with specific fuzzy numbers, it is found that the lack of knowledge is responsible for incorrect performance. To avoid and maintain the uniformity of the fuzzy numbers, an improved operator like adding, scaling, subtracting, multiplication has been derived for generalized trapezoidal (triangular), sigmoidal and parabolic fuzzy numbers.

1 Introduction

The goal of this study is to use unknown, uncertain, and imprecise data to further evaluate the efficiency of industrial processes. It was accompanied by an improvement of the arithmetic operation. Including three sections of the present plan, the following are described:

Trapezoidal (triangular) fuzzy numbers: Since current arithmetic operations take into account the same trust level for different fuzzy numbers and thus this lack of knowledge contributes to the inaccurate result. To overcome this and maintain the uniformity of the fuzzy numbers, we developed a superior arithmetic operator that uses addition, scalar multiplication, subtraction, and multiplication for generalised (triangle) fuzzy numbers. The benefit of the proposed

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Published online: 16 July 2022

operations is that they maintain uniformity of data and give importance to the data. Arithmetical operations on generalized trapezoidal fuzzy numbers and their applications are given by [1] which we will use in our present work. To enhance the usability, in addition to the linear, non-linear arithmetic operation on generalized triangular intuitionistic fuzzy numbers defined by [2] and their applications are given by [3] will enhance the usability of this paper.

Sigmoidal fuzzy numbers: Traditionally, all studies were conducted using probabilistic and binary states. Unfortunately, such theories have failed to incorporate trustworthy knowledge, and as a result of these flaws, probability theory-based tests do not always provide clinicians with useful information. Hence, Inadequate probabilistic approach is to take these integrated uncertainties into account of the data. With contrast to probability theory, approaches to fuzzy logic based set theory [4] give a valuable technique for resolving uncertainties to solve this constraint.

Parabolic fuzzy numbers: Fuzzy set theory has been used as a valuable method to deal with the complex structures in particular, in which the device's relationship may be too difficult to determine accurately. Nevertheless, by using different types of flushing arithmetic we can note that fuzzy logic can yield various simulated efficiencies and outputs, and the fluoridated arithmetic operations are expected. These increasing operations will employ triangular, fuzzy numbers. In addition to probability theory, techniques based on fuzzy set theory provide a valuable methodology for resolving doubts to solve this issue. Somewhat researchers have been working on arithmetic operations in neon numbers for the last couple of years. This is achieved by using the possibility extension theory of Zadeh or its new, expanded, and possible variant, suggested by Klir that takes the so-called required limitations into account. The algebra of ordered fuzzy numbers (OFN) is defined to deal quantitatively with fuzzy inputs in the same way that it does with real numbers [5]. The introduction of a new type of fuzzy number, the Generalized Hexagonal Fuzzy Number, and its applications to the Multi-Criteria Decision-Making Problem (MCDM) by [6].

1.1 Advantage of the Proposed Approach

(a) To avoid performance concerns caused by the same degree of precision with particular fuzzy numbers, arithmetic operations are defined separately for specific types of fuzzy numbers.
(b) Using inaccurate data and fuzzy arithmetic operations, industrial systems are considered to be more effective.
(c) Improve the usability of fuzzy arithmetic operations in a variety of real-world applications, such as optimization and decision-making.

2 Review of Literature

Traditionally, both tests were done based on both probabilistic and conditional assumptions. However, these expectations do not handle trustworthy knowledge and these limitations do not provide clinicians with realistic information because of the chance-based research and consequently, the combined risks are not adequately likely in the results to be taken into account. In addition to formal logic, approaches based on fuzzy set theory provide a valuable technique for handling issues in this context. Fuzzy set theory approaches [4] give a significant technique for resolving uncertainty [7] introduced Petri nets properties, analysis and applications.

Somewhat researchers have been working on arithmetic operations in neon numbers for the last couple of years. The concept of Fuzzy arithmetic was introduced by [8].Then Fuzzy states are given as the basis of Fuzzy reliability by [9]. For better calculations hybrid arithmetic were defined by [10]. Then some conditions are given to these fuzzy numbers and operations are applied on them, to compute these operations Fuzzy arithmetic with requisite constraints are introduced by [11]. Then concept of Fuzzy was again introduced by [12] which makes a substantial reduction in the number of arithmetic transactions. Calculus on fuzzy numbers is a tool for improving fuzzy arithmetic on fuzzy numbers [5]. After that fuzzy sets are defined on intervals to reduce the uncertainties occur before them for which Interval analysis and fuzzy set theory defined by [13] same as crisp set theory to know more about fuzzy sets the concept of cardinality is given by [14]. Then fuzzy numbers are used for optimization techniques to enhance their certainty in accordance of which α-cut fuzzy arithmetic solve some optimization problems by [15]. As fuzzy set theory was introduced on intervals earlier so the arithmetic was also introduced on them for their usability by [16]. As the problems can be of any type so for solving them arithmetic there was a need to define the arithmetic on every type of fuzzy numbers for which Arithmetic on discrete fuzzy numbers also given by [17].In accordance of it the concept of gradual numbers and their use in fuzzy theory was introduced by [18]. After defining the fuzzy arithmetic some properties of it are analysed by [19], investigated the decomposition of fuzzy numbers. In addition to this a value is given to function of fuzzy variables with continuous function by [20]. A special type of fuzzy numbers, LR fuzzy numbers and their parametric forms and arithmetic operations on them are defined by [21]. Till now the fuzzy sets are not applied for solution of uncertain equation with interval and probabilities, the problem of which was solved by [22]. The real-world wavelength division multiplexing (WDM) network design problem was given by [23] in order to design a telecommunications network. Interval-Valued Degrees of Belief: Applications of Interval Computations to Expert Systems and Intelligent Control defined by [24]. After that arithmetical operations on generalized trapezoidal fuzzy numbers and their applications are given by [1] which we will use in our present work. When Multi-objective reliability-redundancy allocation problems were examined these problems are solved using swarm optimization by [25]. To enhance the usability, in addition to the linear, non-linear arithmetic operation on generalized triangular intuitionistic fuzzy numbers defined by [2]. In addition to this entropy based multi criteria decision making method given by [26]. After defining the horizontal membership function of Fuzzy numbers their applications were introduced by [27] and applications of Pentagonal fuzzy numbers are given by [28]. Different equations are solved under uncertainty by [29]. Rule based systems introduced by [30].In addition to the triangular arithmetic operations also applied in generalized sigmoidal fuzzy numbers which are defined by [31]. Fuzzy numbers are symmetrically triangularized by [32]. In addition to the generalized triangular non linear triangular intuitionistic fuzzy numbers and their applications are given by [3] Industrial systems are analysed by using different types of fuzzy numbers by [6, 33]. Computational method for fuzzy arithmetic operations on triangular fuzzy numbers by extension principle given by [34].Hexagonal fuzzy approximation was done by [6]. Then type 2 triangular and trapezoidal membership function generated clustering based by [35]. Ranking of fuzzy numbers are done on the base of relative position and shape characteristics by [36]. New operations based on function principal introduced in [37]. The symmetric triangular approximation of a fuzzy number which preserves the parameter p∈Ps was computed by [32].

3 Preliminaries

This paper provides some basic concepts and fundamental mathematics for fluid model. The fuzzy cuts, convex functions, membership functions, regular fuzzy set, and fuzzy numbers are all explained in detail.

3.1 Basic Concepts of Fuzzy Set Theory

3.1.1 Fuzzy Sets

Crisp sets' essential notions can be extended and generalised using fuzzy sets. The fuzzy package's ability to permit membership grade is a significant element. A fuzzy collection has a membership level ranging between 1 and 0. Membership is not necessary in a Fuzzy set, i.e. members of one Fuzzy group may be members of other Fuzzy groups also. Widely stretched the notation of evaluation set [0,1] (define in / definitely out) to an interval between 0 and 1 with 0.0 being the absolute false and 1.0 representing absolute truth, In the world of discourse, U is represented as a series of ordered pairs (x, μ_p (x)), i.e.

$$P = \{(x, \mu P(x) | x \in U\} \tag{1}$$

where μ_p (x) is the degree of membership of element of x in the fuzzy set P which shows that **x** corresponds to P.

clearly μ_p (x) ∈ [0,1].

3.1.2 Membership Functions

The membership parameter defines the uncertainty and vagueness of the set, regardless of whether the elements are discrete or continuous. For a fuzzy set P, a function, denoted by μ_p which maps **U** to the space M, i.e. μ_p: U → M, is a membership function. Membership is defined as a subset of non-negative real numbers with a finite supremum within the membership range [0, 1].

The three main characteristics of membership traits are as follows:

1. Core—The membership core for certain fuzzy set P is identified as the universe region characterized by full membership for set P. The core contains the universe's element x in such a way that

   $$\mu_p(x) = 1$$

   The core can be an empty set in a fuzzy system.
2. Support—Support for a fuzzy set P is defined as that universe region which is categorized with a non-zero-membership feature of set P. The support includes universe elements such that

   $$\mu_p(x) > 0$$

3. Boundary—The limits of the membership feature to the fuzzy set P are specified as the universe region which contains non-zero membership, but not competitors. In other words, those elements of the universe comprise such that

   $$\mu_p(x) < 1$$

The boundary elements are those that are partially members of the fuzzy set P.

3.1.3 α-Cut of Fuzzy Set (P^α)

α -cut is one of the most important and widely applied principle introduced by Zadeh [26] in fuzzy set theory. α-cut of a fuzzy set $\mathbf{P}^\alpha$ of X is the set in which the membership values in P, is greater than or equal to α.

Take P, be a fuzzy set on X with P (x) ∈ [0,1].
Such that,

$$P = \{(x, P(x)/x \in X\} \tag{2}$$

So,

$$P^{\alpha} = \{x/P(x) \geq \alpha\} \tag{3}$$

3.1.4 Normal Fuzzy Set

When the universal set "U" confirms that a fuzzy set's membership function is unitary, it is considered normal.

3.1.5 Fuzzy Number

A fuzzy number is a further justification from fuzzy sets and mathematics, which is a standard generalisation of real numbers, because it corresponds to a related set of potential values, where each potential value between 0 and 1 has its own strength. This ability is known as the "membership function."

A convex, real line R membership function is regarded as a fuzzy number i.e., if their membership is partially permanent and at least one $x_0 \in \mathbf{U}$ exists, so $\mu_p(x_0) = 1$. The associated member function defined in [p, q] ≠ 0 is specified as.

$$\mu_P(x) = \begin{cases} f(x); x \in (-\infty, p) \\ 1; x = (p, q) \\ g(x) \;\; ; x \in (q, \infty) \end{cases} \tag{4}$$

where f and g are monotonic, continuous, non-decreasing and non-increasing functions from the right and left, and f (x) = 0 for $x \in (-\infty, \omega_1)$ and g (x) = 0 for $x \in (\omega_2, \infty)$.

3.1.6 Fuzzy Number and It's Arithmetic Operations

3.1.6.1 Triangular Fuzzy Number Triangular Fuzzy Number has three variables. So, P = (p, q, r) is said to be triangular fuzzy number when the membership is specified as

$$\mu_P(x) = \begin{cases} 0, & x < p; \\ \frac{x-p}{q-p}, p & p \leq x \leq q \\ \frac{r-x}{r-q}, q & q \leq x \leq r; \\ 0, & x > r \end{cases} \tag{5}$$

The fuzzy numbers' α -cut (p, q, r) is defined and illustrated graphically below.

Whose trust interval is

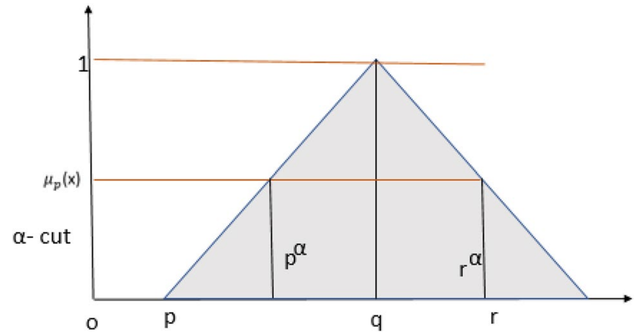

Fig. 1 A graphic representation of the fuzzy triangle number

$$P_{\alpha} = \left[p^{(\alpha)}, r^{(\alpha)}\right] = \left[(q-p)\alpha + p, -(r-q)\alpha + r\right] \tag{6}$$

On two TFNs:$P = (p_1, q_1, r_1)$ and $Q = (p_2, q_2, r_2)$.perform basic arithmetic operations such as addition, subtraction, multiplication, and division (Fig. 1).where $p_i \geq 0$, i = 1,2 are defined as.

1. Addition: $P + Q = (p_1 + p_2, q_1 + q_2, r_1 + r_2)$
2. Subtraction: $P - Q = (p_1 - r_2, q_1 - q_2, r_1 - p_2)$
3. Multiplication: $P \times Q = (p_1\ p_2, q_1\ q_2, r_1\ r_2)$
4. Division: $P \div Q = \left(\frac{p_1}{r_2}, \frac{q_1}{q_2}, \frac{r_1}{p_2}\right)$ if $p_2 > 0$

3.1.6.2 Trapezoidal Membership Function In trapezoidal function there are four variables, P = (x: p, q, r, s), a lower limit is p, upper limit is s and in between there are other two limits q and r, such that p < q < r < s (Fig. 2).

Therefore, its membership is,

$$\mu_P(x) = \begin{cases} 0, & x < p \\ \frac{x-p}{q-p}, p & p \leq x < q \\ 1 & q \leq x < r \\ \frac{s-x}{s-r}, q & r \leq x < s \\ 0, & x \geq r \end{cases} \tag{7}$$

The α − cut of the number P = (p, q, r, s) is the closed interval

$$P\alpha = \left[P^L_{\alpha}, P^R_{\alpha}\right] = \left[p + \alpha(q-p), s - \alpha(s-r)\right], \alpha \in (0, 1] \tag{8}$$

3.1.7 Sigmoidal Fuzzy Numbers

Sigmoidal functions is that computable function $\varphi : R \to R$ is known as sigmoidal function.

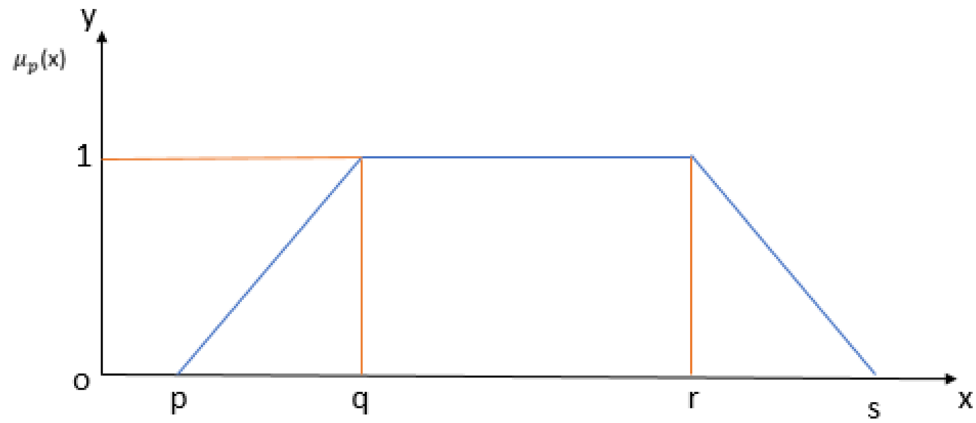

Fig. 2 A graphic representation of the fuzzy trapezoidal number

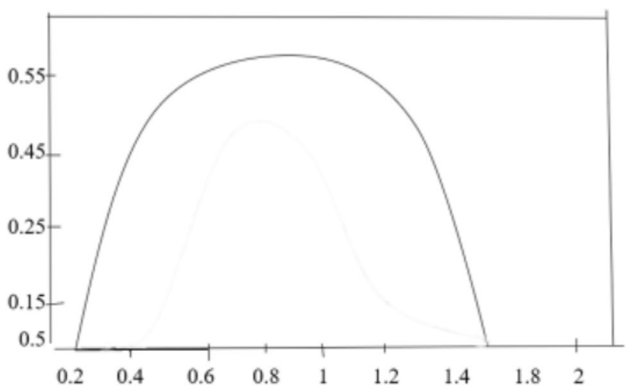

Fig. 3 A graphic representation of the fuzzy sigmoidal number

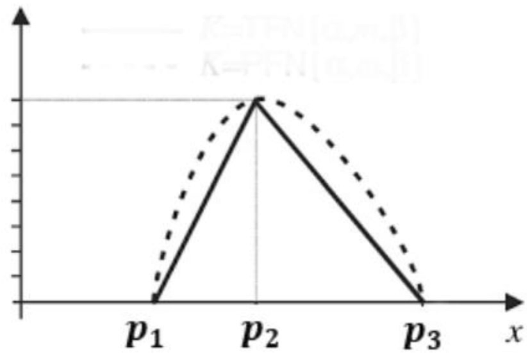

Fig. 4 A graphic representation of the fuzzy parabolic number

each time, $\lim_{x\to-\infty} \varphi(x) = 0$ and $\lim_{X\to+\infty} \varphi(x) = 1$

Such functions typically have S-shaped curves and the following functions are given as standard sigmoidal.

$$\varphi(P) = \frac{1}{1 + e^{-P}} \tag{9}$$

Since, φ (5) = 0.9933 and φ (−5) = 0.0067, covering almost the entire [0, 1] set. In this analysis, the area of this function has therefore been examined [−5, 5], and hence, the membership function related to $P = \left(p_1, p_2, p_3; \varepsilon\right)$ is defined as follows (Fig. 3):

$$\mu_P(x) = \begin{cases} \varepsilon\left(\dfrac{\varphi\left[\left(x - \frac{p_1+p_2}{2}\right)\left(\frac{10}{p_2-p_1}\right)\right] - \varphi(-5)}{\varphi(5) - \varphi(-5)}\right) & ;\ \textit{if}\ p_1 \le x < p_2, \\ \varepsilon & ;\ \textit{if}\ x = p_2, \\ \varepsilon\left(\dfrac{\varphi(5) - \varphi\left[\left(x - \frac{p_2+p_3}{2}\right)\left(\frac{10}{p_3-p_2}\right)\right]}{\varphi(5) - \varphi(-5)}\right) & ;\ \textit{if}\ p_2 \le x < p_3, \\ 0 & ;\ \textit{otherwise}. \end{cases} \tag{10}$$

3.1.8 Parabolic Fuzzy Number

A fuzzy number $A = \left(p_1, p_2, p_3\right)$ is a parabolic fuzzy number if it is described as follows (Fig. 4):

$$\mu_{\overline{A}}(x) = \begin{cases} \left(\frac{x-p_1}{p_2-p_1}\right)^2 & ,\ \text{if}\ p_1 \le x < p_2 \\ 1 & \text{if}\ x = p_2 \\ \left(\frac{p_3-x}{p_3-p_2}\right)^2 & ,\ \text{if}\ p_2 \le x < p_3 \\ 0, & \text{if otherwise} \end{cases} \tag{11}$$

4 Generalized Fuzzy Number

Fuzzy number $sP = \left\langle \left(p_1, p_2, p_3; \varepsilon\right) \mid p_i \in R \right\rangle$, is knows as generalized fuzzy number, if it's membership function i.e. $\mu_P(x) : R \to [0, 1]$ possess the following properties:

1. Continuity.
2. Zero for all $x \in (-\infty, p_1] \cup [p_3, \infty)$.
3. Growing dramatically on $[p_1, p_2]$ and decreasing on $[p_2, p_3]$.

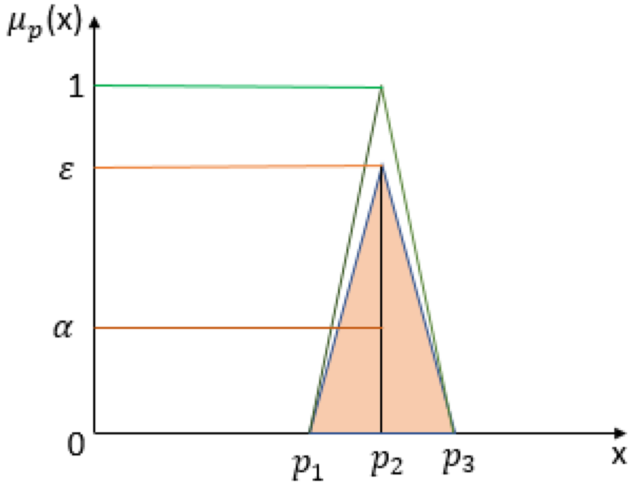

Fig. 5 A graphic representation of the generalized fuzzy triangle number

4. μ_p (x) = ε for all x ∈ p_2; where 0 < ε ≤ 1.

If ε = 1 then P is called normal, otherwise the fuzzy numeral is generalized.

4.1 Generalized Triangular Fuzzy Numbers

If the membership function of a fuzzy number $P = (p_1, p_2, p_3; \varepsilon)$ is defined

$$\mu_P(x) = \begin{cases} \varepsilon\left(\frac{x-p_1}{p_2-p_1}\right) & ; \text{if } p_1 \le x < p_2, \\ \varepsilon & ; \text{if } x = p_2, \\ \varepsilon\left(\frac{p_3-x}{p_3-p_2}\right) & ; \text{if } p_2 \le x < p_3 \\ 0 & ; \text{otherwise} \end{cases} \tag{12}$$

then the fuzzy number P = (p (1,) p (2,) p 3;) is a general triangular fuzzy number (Fig. 5).

The α – cut of the number $P = \langle p_1, p_2, p_3; \varepsilon \rangle$ is the closed interval

$$P\alpha = [P^L_\alpha, P^R_\alpha] = \left[p_1 + \frac{\alpha}{\varepsilon}(p_2 - p_1), p_3 - \frac{\alpha}{\varepsilon}(p_3 - p_2)\right], \alpha \in (0, \varepsilon] \tag{13}$$

4.2 Generalized Trapezoidal Fuzzy Numbers

A fuzzy $P = (p_1, p_2, p_3, p_4; \varepsilon_p)$ is said to be a general trapezoidal fuzzy number, if yhe membership function of P is given by (Fig. 6),

$$\mu_P(x) = \begin{cases} \varepsilon_P\left(\frac{x-p_1}{p_2-p_1}\right) & ; \text{if } p_1 \le x < p_2, \\ \varepsilon_P & ; \text{if } p_2 \le x < p_3; \\ \varepsilon_P\left(\frac{p_4-x}{p_4-p_3}\right) & ; \text{if } p_3 \le x < p_4 \end{cases} \tag{14}$$

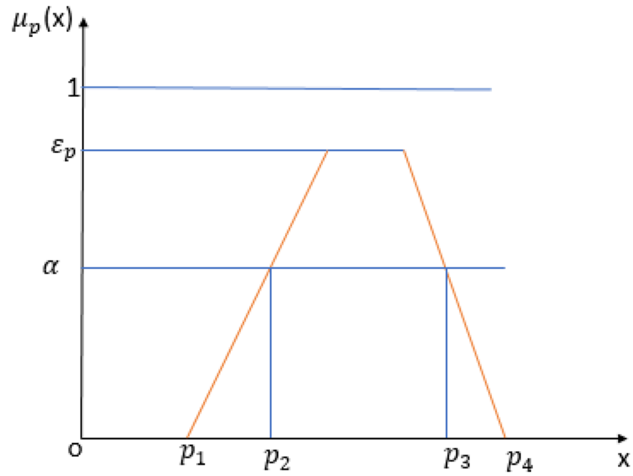

Fig. 6 A graphic representation of the generalized fuzzy trapezoidal number

where p_1, p_2, p_3, p_4 are real numbers, $0 \le \varepsilon_p \le 1$. We quantify improved arithmetical operations on the basis of these α – cuts: addition, subtraction, scalar propagation, division, etc., between the two generalized fuzzy number.

4.2.1 Fuzzy Arithmetic Operations

Let two generalized trapezoidal fuzzy numbers P_1 and P_2 are given by:

$P_1 = (p_{1,} q_{1,} r_{1,} s_1; \varepsilon_1)$ and $P_2 = (p_{2,} q_{2,} r_{2,} s_2; \varepsilon_2)$

Then basic arithmetic operations defined between them are,

1. Addition of P_1 and P_2:

$$P_1 + P_2 = (p_1 + p_{2,} q_1 + q_{2,} r_1 + r_{2,} s_1 + s_2 \ ; \min(\varepsilon_1, \varepsilon_2))$$

2. Subtraction of P_1 and P_2:

$$P_1 - P_2 = (p_1 - s_{2,} q_1 - r_{2,} r_1 - q_{2,} s_1 - p_2 \ ; \min(\varepsilon_1, \varepsilon_2))$$

3. Multiplication of P_1 and P_2:

$$P_1.P_2 = (p_1 p_{2,} q_1 q_{2,} r_1 r_{2,} s_1 s_2 \ ; \min(\varepsilon_1, \varepsilon_2)) \ ; \text{if} p_{1,} p_2 > 0$$

4. Division of P_1 and P_2:

$$\frac{P_1}{P_2} = \left(\frac{p_1}{s_2}, \frac{q_1}{r_2}, \frac{r_1}{q_2}, \frac{s_1}{p_2} \ ; \min(\varepsilon_1, \varepsilon_2)\right)$$

It was noted that some vulnerabilities were shown with the help of the following examples;

4.3 Arithmetic Procedures Among Generalized Fuzzy Numbers

Theorem 1 *By addition of two generalized fuzzy values i.e.* $P = (p_1, p_2, p_3, p_4; \varepsilon_P)$ *and* $Q = (q_1, q_2, q_3, q_4; \varepsilon_Q)$, *a new trapezoidal fuzzy number "R" with two separate confidence levels will be generated*:

$$R = P + Q = (r_1, r_2, r_3, r_4; \varepsilon) \tag{15}$$

where,

$r_1 = p_1 + q_1, r_2 = q_1 + p_2 + \frac{\varepsilon(q_2 - q_1)}{\varepsilon_Q}, r_3 = q_4 + p_3 + \frac{\varepsilon(q_4 - q_3)}{\varepsilon_Q}$ *and* $r_4 = p_4 + q_4$

Proof: Appendix A.

Theorem 2 (Scalar multiplication of fuzzy number).

By multiplication of a scalar number with trapezoidal fuzzy number i.e. $P = (p_1, p_2, p_3, p_4; \varepsilon)$ *a new generalized trapezoidal fuzzy number kP is generated whose value given by.*

$$kP = \begin{cases} (kp_1, kp_2, kp_3, kp_4;\ \varepsilon); if & k > 0 \\ (kp_4, kp_3, kp_2, kp_1;\ \varepsilon); if & k < 0 \end{cases} \tag{16}$$

Proof: Appendix B.

Theorem 3 (Subtraction of two numbers).

Deliberate two generalized fuzzy numbers $P = (p_1, p_2, p_3, p_4; \varepsilon_P)$ *and* $Q = (q_1, q_2, q_3, q_4; \varepsilon_Q)$, *generates a trapezoidal fuzzy number with two separate confidence levels,*

$$R = P - Q = (r_1, r_2, r_3, r_4; \varepsilon) \tag{17}$$

where, $r_1 = p_1 - q_4, r_2 = p_2 - q_4 + \frac{\varepsilon(q_4 - q_3)}{\varepsilon_Q}$, $r_3 = p_3 - q_1 - \frac{\varepsilon(q_2 - q_1)}{\varepsilon_Q}$, *and* $r_4 = p_4 - q_1$

Proof. *We omit this since it follows from Theorems 1 and 2.*

Theorem 4 (Multiplication of two numbers).

$P = (p_1, p_2, p_3, p_4; \varepsilon_P)$ *and* $Q = (q_1, q_2, q_3, q_4; \varepsilon_Q)$ *are two generalised fuzzy numbers with two independent confidence levels such that* $\varepsilon_P \le \varepsilon_q$ *then*

$$R = P \times Q = (r_1, r_2, r_3, r_4; \varepsilon) \tag{18}$$

$\varepsilon = \min(\varepsilon_P, \varepsilon_q)$; *generate a fuzzy numberwhere,* $r_1 = p_1 q_1$, $r_2 = \frac{\varepsilon(p_2 q_2 - p_2 q_1)}{\varepsilon_Q} + p_2 q_1$, $r_3 = \frac{\varepsilon(p_3 q_3 - p_3 q_4)}{\varepsilon_Q} = p_3 q_4$, *and* $r_4 = p_4 q_4$

Proof: Appendix C.

Theorem 5 *For two generalized fuzzy numbers,* $P = (p_1, p_2, p_3, p_4; \varepsilon_P)$ *and* $Q = (q_1, q_2, q_3, q_4; \varepsilon_Q)$, *with two separate confidence levels such that* $\varepsilon_P \le \varepsilon_q$ *then,*

$$R = \frac{P}{Q} = (r_1, r_2, r_3, r_4; \varepsilon) = \text{in}(\varepsilon_P, \varepsilon_q); \text{generate a fuzzy number} \tag{19}$$

where, $r_1 = \frac{p_1}{q_4}$, $r_2 = \frac{\varepsilon\left(\frac{p_2}{q_3} - \frac{p_2}{q_4}\right)}{\varepsilon_Q} + \frac{p_2}{q_4}$, $r_3 = \frac{\varepsilon\left(\frac{p_3}{q_2} - \frac{p_3}{q_1}\right)}{\varepsilon_Q} + \frac{p_3}{q_1}$, *and* $r_4 = \frac{p_4}{q_1}$

Proof.

As, $Q = (q_1, q_2, q_3, q_4; \varepsilon_Q)$. *Thus* $\frac{1}{Q} = \left(\frac{1}{q_4}, \frac{1}{q_3}, \frac{1}{q_2}, \frac{1}{q_1}; \varepsilon\right)$, *and* $\frac{P}{Q} = P \times \left(\frac{1}{Q}\right)$.

As a result, we ignore the proof of this theorem because it follows from Theorem 3.2.4.

4.4 Generalized Parabolic Fuzzy Number

A fuzzy number given by $\tilde{A} = (p_1, p_2, p_3; \varepsilon)$, is generalized parabolic fuzzy when its membership is

$$\mu_{\overline{A}}(x) = \begin{cases} \varepsilon\left(\frac{x - p_1}{p_2 - p_1}\right)^2 & ; \text{ if } p_1 \le x < p_2 \\ \varepsilon & ; \text{ if } x = p_2 \\ \varepsilon\left(\frac{p_3 - x}{p_3 - p_2}\right)^2 & ; \text{ if } p_2 \le x < p_3 \\ 0 & ; \text{ otherwise} \end{cases} \tag{20}$$

Theorem 6 *If X and Y are the two fuzzy Sigmoid numbers that span the globe, their membership is determined as follows*:

$$\mu_X(x) = \begin{cases} \varepsilon_1 L_1(x)\ ; \text{if } p_1 \le x < p_2, \\ \varepsilon_1\ ; \text{if } x = p_2, \\ \varepsilon_1 R_1(x)\ ; \text{if } p_2 \le x < p_3 \\ 0\ ; \text{otherwise} \end{cases} \tag{21}$$

And,

$$\mu_Y(y) = \begin{cases} \varepsilon_1 L_1(y)\ ; \text{if } q_1 \le y < q_2, \\ \varepsilon_1\ ; \text{if } y = q_2, \\ \varepsilon_1 R_1(y)\ ; \text{if } q_2 \le y < q_3 \\ 0\ ; \text{if otherwise} \end{cases} \tag{22}$$

Then there's the fuzzier adjustable Similarly, Z=X+Y is a parabolic fuzzy number with a membership value.

$$\mu_Z(x) = \begin{cases} \varepsilon\left(\frac{x-(p_1+q_1)}{p_2-p_1+q_2-q_1}\right)^2 & ; p_1+q_1 \le x < p_2+q_2 \\ \varepsilon & ; x = p_2+q_2 \\ \varepsilon\left(\frac{(p_3+q_3)-x}{p_3-p_2+q_3-q_2}\right)^2 & ; p_2+q_2 \le x < p_3+q_3 \\ 0 & ; \text{otherwise} \end{cases} \tag{23}$$

Proof: *Appendix D*.

Theorem 7 *Because X is a fuzzy parabolic number and z=kx is the transform, kX is also a sigmoidal fuzzy number given by*:

$$kX = \begin{cases} (kp_1, kp_2, kp_3; \varepsilon_1) \text{ if } k > 0 \\ (kp_3, kp_2, kp_1; \varepsilon_1) \text{ if } k < 0 \end{cases} \tag{24}$$

Proof: Appendix E.

Theorem 8 *If X and Y are the two fuzzy parabolic number of the world's function U. The Z variable given by Z=X–Y is then a parabolic fuzzy value whose membership is,*

$$\mu_Z(x) = \begin{cases} \varepsilon\left(\frac{x-(p_1-q_3)}{p_2-p_1+q_3-q_2}\right)^2; & p_1-q_3 \le x < p_2-q_2 \\ \varepsilon; & x = p_2-q_2 \\ \varepsilon\left(\frac{(p_3-q_1)-x}{p_3-p_2+q_2-q_1}\right)^2; & p_2-q_2 \le x < p_3-q_1 \\ 0; & \text{otherwise} \end{cases} \tag{25}$$

Proof

This proof is insignificant with the use of addition as well as scalar increase ($k = -1$; *which is less than* 0) *of the two parabolic fuzzy numbers*.

Theorem 9 *The two functions of the signature of parabolic membership are given by X and Y, then the variable Z=X.Y, is also a parabolic fuzzy number, which functions as a member, is given by*:

$$\mu_{XY}(x) = \begin{cases} \varepsilon\left(\frac{-Q_1+\sqrt{Q_1^2-4P_1(R_1-x)}}{2P_1}\right)^2 ; & p_1q_1 \le x < p_2q_2 \\ \varepsilon; & x = p_2q_2 \\ \varepsilon\left(\frac{-Q_2+\sqrt{Q_2^2-4P_2(R_2-x)}}{2P_2}\right)^2 ; & p_2q_2 \le x < p_3q_3 \\ 0; & \text{otherwise} \end{cases} \tag{26}$$

where, $P_1 = (p_2-p_1)(q_2-q_1)$, $Q_1 = p_1(q_2-q_1) + b_1(p_2-p_1)$, $R_1 = p_1q_1$

$P_2 = (p_3-p_2)(q_3-q_2)$, $Q_2 = -p_3(q_3-q_2) - q_3(p_3-p_2)$ and $R_2 = p_3q_3$.

Proof: Appendix F.

Theorem 10 *If a fuzzy number* X *denote the parabolic membership which is given in Equation*.

$$\mu_x(x) = \begin{cases} \varepsilon_1 L_1(x); \text{ if } p_1 \le x < p_2, \\ \varepsilon_1; \text{ if } x = p_2 \\ \varepsilon_1 R_1(x); \text{ if } p_2 \le x < p_3 \\ 0; \text{ otherwise} \end{cases} \tag{27}$$

So, the inverse $X^{-1} = [p_3^{-1}, p_2^{-1}, p_1^{-1}; \varepsilon_1]$ *will also be a parabolic fuzzy number whose membership is denoted as*:

$$\mu_{X^{-1}}(x) = \begin{cases} \varepsilon_1\left(\frac{xp_3-1}{x(p_3-p_2)}\right)^2; & \text{if } p_3^{-1} \le x < p_2^{-1} \\ \varepsilon_1; & \text{if } x = p_2^{-1} \\ \varepsilon_1\left(\frac{1-p_1x}{x(p_2-p_1)}\right)^2; & \text{if } p_2^{-1} \le x < p_1^{-1} \\ 0; & \text{otherwise} \end{cases} \tag{28}$$

Proof: Appendix G.

Theorem 11 *X and Y are the* 2 *fuzzy numbers around the World U. If* $0 \notin Y$, *then fuzzy adjustable* $Z = \frac{X}{Y} or X \times Y^{-1}$ *will be also a parabolic fuzzy number*.

Proof

By help of the Theorems 4 and 5, given value will become the membership of $Z = X \times Y^{-1}$

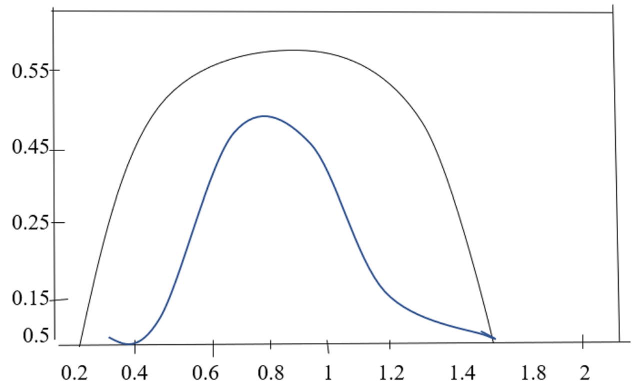

Fig. 7 A graphic representation of the generalized fuzzy sigmoidal number

4.5 Generalized Sigmoidal Fuzzy Numbers

If membership function of a fuzzy number $\tilde{A} = \left(p_1, p_2, p_3; \varepsilon\right)$ related to $P = \left(p_{1,}p_{2,}p_3; \varepsilon\right)$ is defined as follows:

$$\mu_P(x) = \begin{cases} \varepsilon\left(\dfrac{\varphi\left[\left(x - \frac{p_1+p_2}{2}\right)\left(\frac{10}{p_2-p_1}\right)\right] - \varphi(-5)}{\varphi(5) - \varphi(-5)}\right); & \text{if } p_1 \le x < p \\ \varepsilon; & \text{if } x = p2 \\ \varepsilon\left(\dfrac{\varphi(5) - \varphi\left[\left(x - \frac{p_2+p_3}{2}\right)\left(\frac{10}{p_3-p_2}\right)\right]}{\varphi(5) - \varphi(-5)}\right); & \text{if } p_2 \le x < p \\ 0; & \text{otherwise} \end{cases} \tag{29}$$

Then the given fuzzy number will be a sigmoidal fuzzy number.

4.5.1 Membership for Function of a Fuzzy Inconstant

Let function: $R^n \to R$ be a function, and let ξ_1, $\xi_2 \ldots\ldots\ldots \xi_n$, be fuzzy Φ space variables. Then, $\xi = f(\xi_1, \xi_2 \ldots\ldots\ldots \xi_n)$, is a fuzzy variable defined as $\xi(\theta) = f(\xi_1(\theta), \xi_2(\theta) \ldots\ldots\ldots \xi_n(\theta))$, for any $\theta \epsilon \Phi$. If the fuzzy variables which is described in various spaces, then $\xi = f(\xi_1, \xi_2 \ldots\ldots\ldots \xi_n)$, is a fuzzy variable defined on the product space Φ as, $\xi(\theta_1, \theta_2 \ldots\ldots\ldots, \theta_n) = f(\xi_1(\theta_1), \xi_2(\theta_2) \ldots\ldots\ldots \xi_n(\theta_n))$, for any $(\theta_1, \theta_2, \ldots\ldots\ldots, \theta_n)\ \epsilon\Phi$ (Fig. 7).

4.5.2 Fuzzy Arithmetic Operations

Let, $X = \left(p_1, p_2, p_3; \varepsilon_1\right)$ and $Y = \left(q_1, q_2, q_3, ; \varepsilon_2\right)$, be two sigmoid fuzzy numbers with the following membership:

$$\mu_X(x) = \begin{cases} \varepsilon_1 L_1(x); \text{ if } p_1 \le x < p_2 \\ \varepsilon_1; \text{ if } x = p_2 \\ \varepsilon_1 R_1(x); \text{ if } p_2 \le x < p_3 \\ 0; \text{ if otherwise} \end{cases} \tag{30}$$

And,

$$\mu_Y(y) = \begin{cases} \varepsilon_1 L_1(y); \text{ if } q_1 \le y < q_2, \\ \varepsilon_1; \quad \text{if } y = q_2, \\ \varepsilon_1 R_1(y); \text{ if } q_2 \le y < q_3 \\ 0; \quad \text{if otherwise} \end{cases} \tag{31}$$

where,

$$L_1(x) = \frac{\varphi\left[\left(x - \frac{p_1+p_2}{2}\right)\left(\frac{10}{p_2-p_1}\right)\right] - \varphi(-5)}{\varphi(5) - \varphi(-5)} \tag{32}$$

And,

$$L_1(y) = \frac{\varphi\left[\left(y - \frac{q_1+q_2}{2}\right)\left(\frac{10}{q_2-q_1}\right)\right] - \varphi(-5)}{\varphi(5) - \varphi(-5)} \tag{33}$$

are the functions of the left distribution,

$$R_1(x) = \frac{\varphi(5) - \varphi\left[\left(x - \frac{p_2+p_3}{2}\right)\left(\frac{10}{p_3-p_2}\right)\right]}{\varphi(5) - \varphi(-5)} \tag{34}$$

And,

$$R_1(y) = \frac{\varphi(5) - \varphi\left[\left(y - \frac{q_2+q_3}{2}\right)\left(\frac{10}{q_3-q_2}\right)\right]}{\varphi(5) - \varphi(-5)} \tag{35}$$

are the functions of right distribution.

To calculate the distribution functions of the arithmetic operations, we begin by equating L_1 (x) with L_1 (y) and R_1 (x) with R_1 (y), yielding $y = y = \phi_1$ (x) and $y = \phi_2$ (x), respectively, such that,

$$\phi_1(x) = \frac{q_1 + q_2}{2} + \frac{q_2 - q_1}{p_2 - p_1}\left(x - \frac{p_1 + p_2}{2}\right) \tag{36}$$

$$\phi_2(x) = \frac{q_2 + q_3}{2} + \frac{q_3 - q_2}{p_3 - p_2}\left(x - \frac{p_2 + p_3}{2}\right) \tag{37}$$

Let Z be an arithmetic product of X and Y operations. Then at value of $y = \varnothing_1$ (x) and $\varnothing_2$ (x), we get value of $x = \xi_1(z)$ and$\xi_2(z)$, respectively. The function of distribution for fuzzy variable F (z) can be computed on the basis of distribution function of X and Y. Then F (z)=$(z_1, z_2, z_3; \varepsilon)$ where;

$\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ as follows:

$$f_1(x) = \frac{d}{dx} L_1(x) = n_1(z)\, at\, x = \xi_1(z) \tag{38}$$

$$g_1(x) = \frac{d}{dx}R_1(x) = n_2(z)\, at\, x = \xi_2(z) \tag{39}$$

In addition.

$$\frac{dx}{dz} = \frac{d}{dz}\left(\xi_1(z)\right) = m_1(z); \frac{dx}{dz} = \frac{d}{dz}\left(\xi_2(z)\right) = m_2(z)$$

Thus, the membership function of F (z) is given by,

$$\mu_{F(z)}(x) = \begin{cases} \varepsilon \int_{z_1}^{x} n_1(z)m_1(z)dz; \text{if} z_1 \leq x < z_2, \\ \varepsilon; \quad \text{if} x = z_2, \\ \varepsilon \int_{x}^{z_3} n_2(z)m_2(z)dz; \text{if} z_2 \leq x < z_3 \end{cases} \tag{40}$$

Theorem 12 *If the two fuzzy Sigmoid are X and Y across the world U, in order to determine their membership functions is,*

$$\mu_X(x) = \begin{cases} \varepsilon_1 L_1(x) \text{ ; if } p_1 \leq x < p_{2,} \\ \varepsilon_1 \text{ ; if } x = p_{2,} \\ \varepsilon_1 R_1(x) \text{ ; if } p_2 \leq x < p_3 \\ 0 \text{ ; if otherwise} \end{cases} \tag{41}$$

And,

$$\mu_Y(y) = \begin{cases} \varepsilon_1 L_1(y); \text{ if } q_1 \leq y < q_2, \\ \varepsilon_1; \quad \text{if } y = q_2, \\ \varepsilon_1 R_1(y); \text{ if } q_2 \leq y < q_3 \\ 0; \quad \text{if otherwise} \end{cases} \tag{42}$$

then there's the fuzzy adjustable Z=X+Y is a sigmoidal fuzzy with the following membership:

$$\mu_z(Z) = \begin{cases} \varepsilon \frac{\varphi\left[\left(z - \frac{p_1+p_2+q_1+q_2}{2}\right)\left(\frac{10}{p_2+q_2-p_1-q_1}\right)\right] - \varphi(-5)}{\varphi(5)-\varphi(-5)}; if\, p_1 + q_1 \leq z < p_2 + q_2 \\ \varepsilon; if z = p_2 + q_2 \\ \varepsilon \frac{\varphi(5) - \varphi\left[\left(z - \frac{p_2+p_3+q_2+q_3}{2}\right)\left(\frac{10}{p_3+q_3-p_2-q_2}\right)\right]}{\varphi(5)-\varphi(-5)}; if\, p_2 + q_2 \leq z < p_3 + q_3 \\ 0; if otherwise \end{cases} \tag{43}$$

Proof. Appendix H.

Theorem 13 *If X is a fuzzy sigmoidal number and z is the product of scalar number k and element of X equal to kx is the transform, so k is a sigmoidal fuzzy as well given by:*

$$kX = \begin{cases} \left(kp_1, kp_2, kp_3; \varepsilon_1\right) & \text{if} \quad k > 0 \\ \left(kp_3, kp_2, kp_1; \varepsilon_1\right) & \text{if} \quad k < 0 \end{cases} \tag{44}$$

Proof. Appendix I.

Theorem 14 *If X and Y are the two sigmoids of the U world functions, then Z=X–Y is a sigmoidal fuzzy number whose membership is.*

$$\mu_Z(z) = \begin{cases} \varepsilon \frac{\varphi(\{z - \frac{p_1+p_2-q_1-q_2}{2}\} \frac{10}{(p_2-q_2-p_1+q_1)}) - \varphi(-5)}{\varphi(5)-\varphi(-5)}; & p_1 + q_1 \leq z < p_2 + q_2 \\ \varepsilon; & z = p + q_2 \\ \varepsilon \frac{\varphi(5) - \varphi(\{z - \frac{p_2+p_3-q_2-q_3}{2}\} \frac{10}{(p_3-q_3-p_2+q_2)})}{\varphi(5)-\varphi(-5)}; & p_2 + q_2 \leq z < p_3 + q_3 \end{cases} \tag{45}$$

Proof

The evidence is negligible by adding and scaling multipling ($k = -1 < 0$) *of two sigmoidal fuzzy numbers.*

Theorem 15 *When X and Y are two functions of the sigmoidal membership signature, the variable Z equals X. Y is also the sigmoidal fuzzy number, and its function as a member is given by,*

$$\mu_{XY}(z) = \begin{cases} \varepsilon \frac{\varphi(\tau_1) - \varphi(-5)}{\varphi(5)-\varphi(-5)}; & p_1 q_1 \leq z < p_2 q_2 \\ \varepsilon; & z = p_2 q_2 \\ \varepsilon \frac{\varphi(5) - \varphi(\tau_2)}{\varphi(5)-\varphi(-5)}; & p_2 q_2 \leq z < p_3 q_3 \end{cases} \tag{46}$$

where, $\tau_1 = 10\left(\frac{-Q_1 + \sqrt{Q_1^2 - 4P_1(R_1 - z)} - P_1}{2P_1}\right)$, $\tau_2 = 10\left(\frac{-Q_2 - \sqrt{Q_2^2 - 4P_2(R_2 - z)} + P_2}{2P_2}\right)$,

$P_1 = (p_2 - p_1)(q_2 - q_1)$, $Q_1 = p_1(q_2 - q_1) + q_1(p_2 - p_1)$, $R_1 = p_1 q_1$

$P_2 = (p_3 - p_2)(q_3 - q_2)$, $Q_2 = -p_3(q_3 - q_2) - q_3(p_3 - p_2)$, *and* $R_2 = p_3 q_3$

Proof. Appendix J.

Theorem 16 *Proof*

If X and Y are the different fuzzy numbers that orbit the Universe U, then 0 *belongs to Y, the fuzzy adjustable* $Z = \frac{X}{Y} = X \cdot Y^{-1}$ *is a sigmoidal fuzzy number as well.*

By means of the theorem 4, *we can come to the membership function*, $Z = X \cdot Y^{-1}$

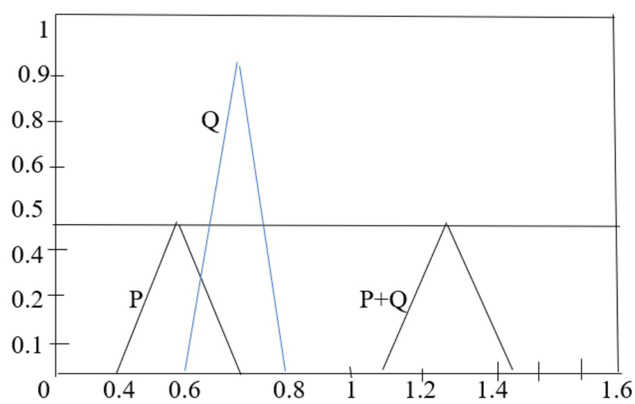

Fig. 8 A graphic representation the addition of the two generalised fuzzy triangle number

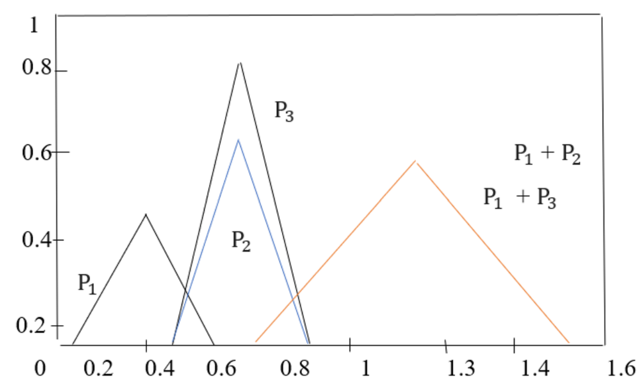

Fig. 9 A graphic representation the addition of generalised fuzzy triangle number

5 Application of Basic Fuzzy Number

5.1 Trapezoidal (Triangular) Fuzzy Numbers

Example 1 Consider the generalised triangle fuzzy numbers, P = (0.5, 0.6, 0.7; 0.5) and Q = (0.6, 0.7, 0.8; 0.9), as illustrated in the figure below. After the above-mentioned numbers are added we get, P + Q = (1.1,1.3,1.6;0.5) and therefore, the resulting number is a generalized fuzzy triangular number. The figure, however, demonstrated that if we take a 0.5 (= min (0.5, 0.9)) break from Q, the Fuzzy Q is then transformed it into generalised trapezoidal fuzzy number. Thus, in Chen's operations the triangular fuzzy number changes into the trapezoidal fuzzy number, and thus the smoothness of the figures are not preserved. Hence, this flatness must be preserved in the widespread fuzzy number. Therefore, the current method loses its value (Fig. 8).

Example 2 As shown in Figure, $P_1 = (0.2, 0.4, 0.6; 0.5)$, $P_2 = (0.5, 0.7, 0.9; 0.7)$, and P 3 = (0.5, 0.7, 0.9; 0.9) are generalised triangular fuzzy numbers. It can be seen from it that if $P_2 \in P_3$, then we have $P_1 + P_2 \in P_1 + P_3$. If, however, if we use the Chen method, we have $P_1 + P_2 = (0.7,1,1.1;0.5)$ and $P_1 + P_3 = (0.7,1,1.1;0.5)$. Thus $P_1 + P_2 \equiv P_1 + P_3$ which breaches the fact $P_1 + P_2 \in P_1 + P_3$. Therefore, the arithmetic operations between general fuzzy numbers cannot be determined using a Chen method (Fig. 9).

Example 3 Length of the Rod Assume the length of the rod is a fuzzy number which is sigmoidal, P = (12, 13.5, 15 cm; 0.8). When the sigmoidal fuzzy number Q = (5, 6.5, 8 cm; 0.7) is cut from the rod, the rod length R = P−Q remains constant. Then, − = (−8, 6.5, 5 cm; 0.7) have a fuzzy number negative Q, And the accompanying sigmoidal representation to P and –Q is:

$$\mu_P(x) = \begin{cases} 0.8\left(\dfrac{\varphi\left(\left\{x-\frac{25.5}{2}\right\}\frac{1.0}{15}\right)-\varphi(-5)}{\varphi(5)-\varphi(-5)}\right); 12 \le x < 13.5 \\ 0.8; \quad x = 13.5 \\ 0.8\left(\dfrac{\varphi(5)-\varphi\left(\left\{x-\frac{28.5}{2}\right\}\frac{10}{1.5}\right)}{\varphi(5)-\varphi(-5)}\right); 13.5 \le x < 15 \\ 0; \quad \text{otherwise} \end{cases} \tag{47}$$

And

$$\mu_{-Q}(y) = \begin{cases} 0.7\left(\dfrac{\varphi\left(\left\{y+\frac{14.5}{2}\right\}\frac{10}{1.5}\right)-\varphi(-5)}{\varphi(5)-\varphi(-5)}\right); \quad -8 \le y < -6.5 \\ 0.7; \quad y = -6.5 \\ 0.7\left(\dfrac{\varphi(5)-\varphi\left(\left\{y+\frac{11.5}{2}\right\}\frac{1.0}{15}\right)}{\varphi(5)-\varphi(-5)}\right); \quad -6.5 \le y < -5 \\ 0; \quad \text{otherwise} \end{cases} \tag{48}$$

As a result, the additional property of the two sigmoidal numbers is employed, and the remaining rod length is a sigmoidal fuzzy number with R membership degree, denoted as:

$$\mu_R(z) = \begin{cases} 0.7\left(\dfrac{\varphi\left(\left\{z-\frac{11}{2}\right\}\frac{10}{3}\right)-\varphi(-5)}{\varphi(5)-\varphi(-5)}\right); \quad 4 \le z < 7 \\ 0.7; \quad z = 7 \\ 0.7\left(\dfrac{\varphi(5)-\varphi\left(\left\{z-\frac{17}{2}\right\}\frac{10}{3}\right)}{\varphi(5)-\varphi(-5)}\right); \quad 7 \le z < 10. \\ 0; \quad \text{otherwise} \end{cases} \tag{49}$$

Example 4 Length of the Rectangle The rectangle's area and width should be given as fuzzy sigmoidal numbers $P = (1,2,4\text{cm}^2; 0.75)$ and Q = (3,5,6 cm; 0.85), respectively, and the rectangle length is characterised as $P(\div)Q$ or $P(\cdot)Q^{-1}$. We now obtain the membership function based on Q for $Q^{-1} = \left(6^{-1}, 5^{-1}, 3^{-1}\text{cm}^{-1};0.85\right)$ is:

$$\mu_{Q^{-1}}(y)=\begin{cases} 0.85\left(\dfrac{\varphi(5)-\varphi\left(\left\{\frac{1}{y}-\frac{11}{2}\right\}10\right)}{\varphi(5)-\varphi(-5)}\right); & 6^{-1} \le y < 5^{-1} \\ 0.85; & y = 5^{-1} \\ 0.85\left(\dfrac{\varphi\left(\left\{\frac{1}{y}-\frac{8}{2}\right\}5\right)-\varphi(-5)}{\varphi(5)-\varphi(-5)}\right); & 5^{-1} \le y < 3^{-1} \\ 0; & \text{otherwise} \end{cases} \tag{50}$$

As a result, the membership function of rectangle length is obtained by combining two sigmoidal fuzzy numbers, P and Q^{-1}. From membership feature, it was inferred that there is a 75% possibility of 0.4 cm length of the rectangle, and its range is $[\frac{1}{6}, \frac{4}{3}]$.

From the information, we may deduce that the rod will be between 4 and 10 cm long, with a 70% chance of being 7 cm long.

Example 5 Length of the Rod Assume that the rod's length is a fuzzy parabolic number, P = (12,13.5,15 cm; 0.8). When the rod is cut to the length Q = (5,6.5,8 cm; 0.7), the rod length remains R = P-Q.The parabolic component function for fluorescent numbers P and Q are described below

$$\mu_P(x) = \begin{cases} 0.8\left(\dfrac{x-12}{1.5}\right)^2; \text{ if } 12 \le x < 13.5 \\ 0.8\left(\dfrac{15-x}{1.5}\right)^2; \text{ if } 13.5 \le x < 15 \\ 0; \quad \text{otherwise} \end{cases} \tag{51}$$

And

$$\mu_Q(x) = \begin{cases} 0.7\left(\dfrac{y-5}{1.5}\right)^2; \text{ if } 5 \le y < 6.5 \\ 0.7\left(\dfrac{8-y}{1.5}\right)^2; \text{ if } 6.5 \le y < 8 \\ 0; \quad \text{otherwise} \end{cases} \tag{52}$$

then, $-Q = (-8, -6.5, -5 \text{ cm}; 0.7)$ is a negative fuzzy number –Q. And the accompanying sigmoidal representation is

$$\mu_{-Q}(y) = \begin{cases} 0.7\left(\dfrac{y+8}{1.5}\right)^2; & \text{if } -8 \le y < -6.5 \\ 0.7; & \text{if } \quad y = -6.5 \\ 0.7\left(\dfrac{y+5}{1.5}\right)^2; & \text{if } -6.5 \le y < -5 \\ 0; & \text{otherwise} \end{cases} \tag{53}$$

Therefore, the membership function of the remaining length of rod is the parabolic fuzzy number R by adding the two parabolic fuzzy numbers and specified as:

$$\mu_R(y) = \begin{cases} 0.7\left(\dfrac{x-4}{3}\right)^2; & \text{if } 4 \le x < 7 \\ 0.7; & \text{if } \quad x = 7 \\ 0.7\left(\dfrac{10-x}{3}\right)^2; & \text{if } 7 \le x < 10 \\ 0; & \text{otherwise} \end{cases} \tag{54}$$

We can deduce from the above that the rest of the rod is between 4 and 10 cm thick.

In addition, at a non-linear rate of $14/9(x-4)$ the length has been raised from 4 to 7 cm and subsequently decreased from 7 to 10 cm at a non-linear rate of decrease $14/9(10-x)$. There are also 70% possibilities for a value of 7 cm in length.

Example 6 Length of the Rectangle The rectangle's area and width are fuzzy sigmoidal numbers P = (1,2,4cm²; 0.75) and Q = (3,5,6 cm; 0.85) respectively, and the rectangle length is $P(\div)Q$ or $P(\cdot)Q^{-1}$. We now obtain the membership function based on Q membership of $Q^{-1} = (6^{-1}, 5^{-1}, 3^{-1}; 0.85)$ is:

$$\mu_{Q^{-1}}(y) = \begin{cases} 0.85\left(6-\dfrac{1}{y}\right)^2; & \text{if } 6^{-1} \le y < 5^{-1} \\ 0.85; & \text{if } \quad y = 5^{-1} \\ 0.85\left(\dfrac{\frac{1}{y}-3}{2}\right)^2; & \text{if } 5^{-1} \le y < 3^{-1} \\ 0; & \text{otherwise} \end{cases} \tag{55}$$

Therefore, by multiplying the two fuzzy numbers P and $Q.^{-1}$, the membership function of the rectangle length is achieved as

$$\mu_{P.Q^{-1}}(y) = \begin{cases} 0.75\left(\dfrac{6x-1}{x+1}\right)^2; & \text{if } \dfrac{1}{6} \le x < \dfrac{2}{5} \\ 0.75; & \text{if } x = \dfrac{2}{5} \\ 0.75\left(\dfrac{4-3x}{2(x+1)}\right)^2; & \text{if } \dfrac{2}{5} \le x < \dfrac{4}{3} \\ 0; & \text{otherwise} \end{cases} \tag{56}$$

It was inferred from the membership feature that 75% were likely to have a rectangle length of 0.4 cm and the range of the rectangle's length is $\left[\frac{1}{6}, \frac{4}{3}\right]$.

6 Conclusion

On the basis of dispersion and associated functions, we worked on generalised (triangular), sigmoidal, and parabolic fuzzy numbers and offered the associated flush arithmetic activities, like addition, subtraction, multiplication, reversal, division, and so on. In our everyday lives, because of human mistakes or other inevitable variables it is impossible to tell the right actions, it is hard to denote or gather the information correctly. To answer this issue, the effects of uncertainties in the data were evaluated by a nonlinear sigmoid logistic function. Concerning decussated values of linear and parabolic membership functions, the COG method measured them and found that preservation should be grounded on defuzzied values instead of crisp values to increase efficiency, as a harmless period is checked earlier the crisp value is hit.

Also, a validity check was performed using generalized parabolic fuzzy numbers to overcome those estimation problems. Specific reliability parameters of different spreads proposed by decision makers were then determined using an improved arithmetic operation and thus their output compared with other current operations. As an additional study, we suggest the development of new efficient numerical methods for the estimation of the basis on the extension principle as well as the appropriate adoption of the proposed generalised fuzzy member.

Appendix A

Proof—Assume P and Q are two generalised fuzzified values with varying levels of confidence, such that $\varepsilon_P < \varepsilon_Q$. Take $\varepsilon = \min(\varepsilon_P, \varepsilon_Q)$ i.e. $\varepsilon = \varepsilon_P$ then α− cut of P and Q are.

$$P_\alpha = \left[p_1 + \alpha\left(\frac{p_2 - p_1}{\varepsilon_P}\right), p_4 - \alpha\left(\frac{p_4 - p_3}{\varepsilon_P}\right)\right]; \forall_\alpha \in [0, \varepsilon_P] s.t. 0 \le \varepsilon_P \le 1$$

$$Q_\alpha = \left[q_1 + \alpha\left(\frac{q_2^* - q_1}{\varepsilon}\right), q_4 - \alpha\left(\frac{q_4 - q_3^*}{\varepsilon}\right)\right]; \forall_\alpha \in [0, \varepsilon] s.t. 0 \le \varepsilon \le 1$$

Let, R=P+Q={x | x ∈ R_α} for all α ∈ [0, ε]. Here $R_\alpha = [R^L(\alpha), R^U(\alpha)]$ be its α− cuts such that, $R^L(\alpha) = P^L(\alpha) + Q^L(\alpha)$, and $R^U(\alpha) = P^U(\alpha) + Q^U(\alpha)$ i.e.

$$R_\alpha = \left[P^L(\alpha) + Q^L(\alpha), P^U(\alpha) + Q^U(\alpha)\right] \tag{57}$$

$$= \left[p_1 + \alpha\left(\frac{p_2 - p_1}{\varepsilon_P}\right) + q_1 + \alpha\left(\frac{q_2^* - q_1}{\varepsilon}\right), p_4 - \alpha\left(\frac{p_4 - p_3}{\varepsilon_P}\right) + q_4 - \alpha\left(\frac{q_4 - q_3^*}{\varepsilon}\right)\right]$$

$$= \left[p_1 + q_1 + \alpha\left(\frac{p_2 - p_1}{\varepsilon_P} + \frac{q_2^* - q_1}{\varepsilon}\right), p_4 + q_4 - \alpha\left(\frac{p_4 - p_3}{\varepsilon_P} + \frac{q_4 - q_3^*}{\varepsilon}\right)\right]$$

Now,
$p_1 + q_1 + \alpha\left(\frac{p_2 - p_1}{\varepsilon_P} + \frac{q_2^* - q_1}{\varepsilon}\right) - x = 0$ and

$$p_4 + q_4 - \alpha\left(\frac{p_4 - p_3}{\varepsilon_P} + \frac{q_4 - q_3^*}{\varepsilon}\right) - x = 0$$

As a result, R's left and right membership functions are.

$$f\frac{L}{R}(x) = \frac{x - p_1 - q_1}{\frac{p_2 - p_1}{\varepsilon_P} - \frac{q_2^* - q_1}{\varepsilon}} \tag{58}$$

$$f\frac{L}{R}(x) = \frac{p_4 + {}_4 - x}{\frac{p_4 - p_3}{\varepsilon_P} + \frac{q_4 - q_3^*}{\varepsilon}} \tag{59}$$

Since $\varepsilon = \varepsilon_P$ and $q_2^* = q_1 + \varepsilon\left(\frac{q_2 - q_1}{\varepsilon_Q}\right)$ and $q_3^* = q_4 - \varepsilon\left(\frac{q_4 - q_3}{\varepsilon_Q}\right)$, Thus, above f_R^L and f_R^R becomes,

$$f\frac{L}{R}(x) = \frac{x - p_1 - q_1}{\frac{p_2 - p_1}{\varepsilon_P} - \frac{q_2^* - q_1}{\varepsilon}}$$

$$= \varepsilon\left(\frac{x - p_1 - q_1}{p_2 - p_1 - q_1 + q_2^*}\right)$$

$$= \varepsilon\left(\frac{x - (p_1 + q_1)}{p_2 + q_1 + \varepsilon\left(\frac{q_2 - q_1}{\varepsilon_Q}\right) - (p_1 + q_1)}\right) \tag{60}$$

For, $p_1 + q_1 \le x \le p_2 + q_1 + \varepsilon\left(\frac{q_2-q_1}{\varepsilon_Q}\right)$Similarly,

$$f_R^R(x) = \varepsilon\left(\frac{(p_4+q_4)-x}{q_4+p_3-\varepsilon\left(\frac{q_2-q_3}{\varepsilon_Q}\right)-(p_4+q_4)}\right) \tag{61}$$

For, $q_4 + p_3 - \varepsilon\left(\frac{q_2-q_3}{\varepsilon_Q}\right) \le x \le (p_4+q_4)$

Thus, adding two generalized fuzzy values is another generalized fuzzy, whose membership function is explained as,

$$x \in \left[p_1 + \alpha\left(\frac{p_2-p_1}{\varepsilon_p}\right), p4 - \alpha\left(\frac{p_4-p_3}{\varepsilon_p}\right)\right]$$

$$y = kx \in \left[kp_1 + \alpha\left(\frac{kp_2-kp_1}{\varepsilon_P}\right), kp_4 - \alpha\left(\frac{kp_4-kp_3}{\varepsilon_P}\right)\right] \tag{63}$$

As a conclusion, the scalar product's membership function is:

$$\mu_R(x) = \begin{cases} \varepsilon\left(\dfrac{x-(p_1+q_1)}{p_2+q_1+\varepsilon\left(\frac{q_2-q_1}{\varepsilon_Q}\right)-(p_1+q_1)}\right); & p_1+q_1 \le x \le p_2+q_1+\varepsilon\left(\dfrac{q_2-q_1}{\varepsilon_Q}\right) \\ \varepsilon; & p_2+q_1+\varepsilon\left(\dfrac{q_2-q_1}{\varepsilon_Q}\right) \le x \le q_4+p_3-\varepsilon\left(\dfrac{q_2-q_3}{\varepsilon_Q}\right) \\ \varepsilon\left(\dfrac{(p_4+q_4)-x}{q_4+p_3-\varepsilon\left(\frac{q_2-q_3}{\varepsilon_Q}\right)-(p_4+q_4)}\right); & q_4+p_3-\varepsilon\left(\dfrac{q_2-q_3}{\varepsilon_Q}\right) \le x \le (p_4+q_4) \\ 0; & \text{otherwise} \end{cases} \tag{62}$$

To put in another way, that's the addition of two generalised fuzzy numbers, given by:

$R = P + Q = (r_1, r_2, r_3, r_4; \varepsilon) = \min(\varepsilon_P, \varepsilon_Q)$ is a generalized fuzzy number where,

$$r_1 = p_1 + q_1$$

$$r_2 = q_1 + p_2 + \frac{\varepsilon(q_2-q_1)}{\varepsilon_Q}$$

$$r_3 = q_4 + p_3 - \frac{\varepsilon(q_4-q_3)}{\varepsilon_Q}$$

$$r_4 = p_4 + q_4$$

Appendix B

Proof—When $k > 0$, the $\alpha -$cut for P's membership function is.

$P_\alpha = [p_1+\alpha(\frac{p_2-p_1}{\varepsilon_P}),\ p_4-\alpha(\frac{p_4-p_3}{\varepsilon_P})]$; $\forall_\alpha \in [0, \varepsilon_P]$ s.t. $0 \le \varepsilon_P \le 1$

This is,therefore,

$$\mu_{kP}(x) = \begin{cases} \varepsilon_P\left(\dfrac{x-kp_1}{kp_2-kp_1}\right); & \text{if } kp_1 \le x \le kp_2 \\ \varepsilon_P; & \text{if } kp_2 \le x \le kp_3 \\ \varepsilon_P\left(\dfrac{kp_4-x}{kp_4-kp_3}\right); & \text{if } kp_3 \le x \le kp_4 \\ 0; & \text{otherwise} \end{cases}$$

Appendix C

Proof—Because there are two distinct fuzzy numbers with confidence levels, ε_P and ε_q defined as $\varepsilon_P \le \varepsilon_q$. So, first of all we'll turn the fuzzy Q into $Q^* = Q = (q_1, q_2^*, q_3^*, q_4; \varepsilon$, where.

$q_2^* = q_1 + \varepsilon\left(\frac{q_2-q_1}{\varepsilon_Q}\right)$and $q_3^* = q_4 - \varepsilon\left(\frac{q_4-q_3}{\varepsilon_Q}\right)$, Now the $\alpha -$cuts that suit P and Q* are equivalent to.

$$P_\alpha = \left[p_1 + \alpha\left(\frac{p_2-p_1}{\varepsilon_P}\right), p_4 - \alpha\left(\frac{p_4-p_3}{\varepsilon_P}\right)\right] \quad ; \quad \forall_\alpha \in [0, \varepsilon_P]\ s.t.\ 0 \le \varepsilon_P \le 1$$

$$Q_\alpha^* = \left[q_1 + \alpha\left(\frac{q_2^*-q_1}{\varepsilon}\right), q_4 - \alpha\left(\frac{q_4-q_3^*}{\varepsilon}\right)\right] \quad ; \quad \forall_\alpha \in [0, \varepsilon_P]\ s.t.\ 0 \le \varepsilon \le 1$$

Suppose $R = P \times Q = \{x | x \in R_\alpha\}$ for all $\alpha \in [0, \varepsilon]$. Here $R_\alpha = [R^L_\alpha, R^U_\alpha]$ be its $\alpha-$cuts such that $R^L_\alpha = P^L_\alpha Q^{*L}_\alpha$ and $R^U_\alpha = P^U_\alpha Q^{*U}_\alpha$ i.e.

$$R_\alpha = [P^L_\alpha Q^{*L}_\alpha, P^U_\alpha Q^{*U}_\alpha]$$

$$= \left[\left\{p_1 + \alpha\left(\frac{p_2 - p_1}{\varepsilon_P}\right)\right\}, \left\{q_1 + \alpha\left(\frac{q^*_2 - q_1}{\varepsilon}\right)\right\}, \left\{p_4 - \alpha\left(\frac{p_4 - p_3}{\varepsilon_P}\right)\right\}\left\{q_4 - \alpha\left(\frac{q_4 - q^*_3}{\varepsilon}\right)\right\}\right]$$

$$= \left[p_1 q_1 + \alpha\left(\frac{p_1(q^*_2 - q_1) + q_1(p_2 - p_1)}{\varepsilon}\right) + \frac{\alpha^2}{\varepsilon^2}(p_2 - p_1)(q^*_2 - q_1)\right]$$

$$\left[p_4 q_4 + \alpha\left(\frac{p_4(q_4 - q^*_3) + q_1(p_4 - p_3)}{\varepsilon}\right) + \frac{\alpha^2}{\varepsilon^2}(p_4 - p_3)(q_4 - q^*_3)\right]$$

Therefore,

$$p_1 q_1 + \alpha\left(\frac{p_1(q^*_2 - q_1) + q_1(p_2 - p_1)}{\varepsilon}\right) + \frac{\alpha^2}{\varepsilon^2}(p_2 - p_1)(q^*_2 - q_1) - x = 0, \text{ and}$$

$$p_4 q_4 + \alpha\left(\frac{p_4(q_4 - q^*_3) + q_1(p_4 - p_3)}{\varepsilon}\right) + \frac{\alpha^2}{\varepsilon^2}(p_4 - p_3)(q_4 - q^*_3) - x = 0$$

Which are the quadratic functions in the α and therefore its roots give the left as well as right membership of R,

$$f^L_R(x) = \frac{-N_1 + \sqrt{N_1^2 + 4M_1(x - J_1)}}{2M_1}; r_1 \le x \le r_2 \tag{64}$$

$$f^U_R(x) = \frac{-N_2 + \sqrt{N_2^2 + 4M_2(x - J_2)}}{2M_2}; r_3 \le x \le r_4 \tag{65}$$

Where, $N_1 = \frac{p_1(q^*_2 - q_1) + q_1(p_2 - p_1)}{\varepsilon}$, $N_2 = \frac{p_4(q_4 - q^*_3) + q_1(p_4 - p_3)}{\varepsilon}$, $M_1 = \frac{(p_2 - p_1)(q^*_2 - q_1)}{\varepsilon^2}, M_2 = \frac{(p_4 - p_3)(q_4 - q^*_3)}{\varepsilon^2}, J_1 = p_1 q_1, J_2 = p_4 q_4$

By Substituting the value of q^*_2 and q^*_3, we get.

$$N_1 = \frac{(p_2 - p_1)(q_2 - q_1)}{\varepsilon\varepsilon_Q}; N_2 = \frac{(p_4 - p_3)(q_4 - q_3)}{\varepsilon\varepsilon_Q} \tag{66}$$

$$M_1 = \frac{p_1(q_2 - q_1)}{\varepsilon_Q} + \frac{q_1(p_2 - p_1)}{\varepsilon}; M_2 = \frac{p_4(q_4 - q_3)}{\varepsilon_Q} + \frac{q_4(p_4 - p_3)}{\varepsilon} \tag{67}$$

As an outcome, the product of two generalised fuzzy numbers is a fuzzy number with the membership,

$$\mu_R(x) = \begin{cases} \frac{-N_1 + \sqrt{N_1^2 + 4M_1(x - J_1)}}{2M_1} & ; \quad r_1 \le x \le r_2 \\ \varepsilon & ; \quad r_2 \le x \le r_3 \\ \frac{-N_2 + \sqrt{N_2^2 + 4M_2(x - J_2)}}{2M_2} & ; \quad r_3 \le x \le r_4 \end{cases} \tag{68}$$

where,

$$r_1 = p_1 q_1$$

$$r_2 = \frac{\varepsilon(p_2 q_2 - p_2 q_1)}{\varepsilon_Q} + p_2 q_1$$

$$r_3 = \frac{\varepsilon(p_3 q_3 - p_3 q_4)}{\varepsilon_Q} + p_3 q_4$$

$$r_4 = p_4 q_4$$

Appendix D

Proof—Consider the membership of two parabolic fuzzy numbers, X and Y be.

$$\mu_X(x) = \begin{cases} \varepsilon_1 L_1(x); \text{ if } p_1 \le x < p_2, \\ \varepsilon_1; \text{ if } x = p_2, \\ \varepsilon_1 R_1(x); \text{ if } p_2 \le x < p_3 \\ 0; \quad \text{if otherwise} \end{cases}$$

And,

$$\mu_Y(y) = \begin{cases} \varepsilon_1 L_1(y); \quad \text{if } q_1 \le y < q_2, \\ \varepsilon_1; \quad \text{if } y = q_2, \\ \varepsilon_1 R_1(y); \quad \text{if } q_2 \le y < q_3 \\ 0; \quad \text{if otherwise} \end{cases}$$

To add the fuzzy numbers X and Y, the next fuzzy number of them, Z = X + Y

Implies $Z = [p_1 + q_1, p_2 + q_2, p_3 + q_3]$

As $z = x + y$ at $y = \varphi_1(x)$ and $\varphi_2(x)$ we get,

$z = x + \varphi_1(x)$ and $z = x + \varphi_2(x)$ respectively.that specifies that $x = \varphi_1(z)$ and $x = \varphi_2(z)$ where

$$x = \varphi_1(z) = \frac{z - \frac{p_2 q_1}{p_2 - p_1} + \frac{p_1 q_2}{p_2 - p_1}}{1 + \frac{(q_2 - q_1)}{p_2 - p_1}} \tag{69}$$

Hence, $\eta_1(z) = \left(\frac{2}{(p_2-p_1)^2}\right)\left(\frac{z-p_1-q_1}{1+\frac{(q_2-q_1)}{p_2-p_1}}\right), m_1(z) = 1 + \frac{(q_2-q_1)}{p_2-p_1}$

so, the left sided distribution function

$$\int_{p_1+q_1}^{X} \eta_1(z) m_1(z) dz = \int_{p_1+q_1}^{X} \left(\frac{2}{(p_2-p_1)^2}\right)\left(\frac{z-p_1-q_1}{1+\frac{(q_2-q_1)}{p_2-p_1}}\right) \times \left(\frac{1}{1+\frac{(q_2-q_1)1}{p_2-p_1}}\right) dz$$

$$= \left(\frac{2}{(p_2-p_1)^2}\right)\left((\frac{1}{1+\frac{(q_2-q_1)}{p_2-p_1}})^2 \times \int_{p_1+q_1}^{X} (z-p_1-q_1\right) dz = (\frac{1}{p_2-p_1})^2 (\frac{x-(p_1+q_1)}{1+\frac{(q_2-q_1)}{p_2-p_1}})^2$$

$$= (\frac{x-(p_1+q_1)}{p_2+q_2-p_1-q_1})^2 ; p_1+q_1 \le x < p_2+q_2$$

Similarly, if $y = \phi_2(x)$ then $z = x + y$ becomes $x = \varphi_2(z)$ where

$$\varphi_2(x) = \frac{z + \frac{-p_2 q_3}{p_3 - p_2} + \frac{p_3 q_2}{p_3 - p_2}}{1 + \frac{q_3 - q_2}{p_3 - p_2}} \tag{70}$$

Here, in this case

$$\eta_2(x) = \left(\frac{-2}{(p_3-p_2)^2}\right)\left(\frac{p_3+q_3-z}{1+\frac{(q_3-q_2)}{p_3-p_2}}\right) \tag{71}$$

so, the right sided distribution function,

$$\int_{p_3+q_3}^{X} \eta_2(z) m_2(z) dz = \int_{p_3+q_3}^{X} \left(\frac{-2}{(p_3-p_2)^2}\right)\left(\frac{p_3+q_3-z}{1+\frac{(q_3-q_2)}{p_3-p_2}}\right) \times \left(\frac{1}{1+\frac{(q_3-q_2)}{p_3-p_2}}\right) dz \tag{72}$$

$$= \left(\frac{-2}{(p_3-p_2)^2}\right)\left((\frac{1}{1+\frac{(q_3-q_2)}{p_3-p_2}})^2 \times \int_{p_3+q_3}^{X} (p_3+q_3-z\right) dz$$

$$= \left(\frac{-2}{(p_3-p_2)^2}\right)(\frac{1}{1+\frac{(q_3-q_2)}{p_3-p_2}})^2 \times \frac{(p_3+q_3-x)^2}{-2}$$

$$= (\frac{1}{p_3-p_2})^2 (\frac{(p_3+q_3)-x}{1+\frac{(q_3-q_2)}{p_3-p_2}})^2$$

$$= (\frac{(p_3+q_3)-x}{p_3-p_2+q_3-q_2})^2 ; p_2+q_2 \le x < p_3+q_3 \tag{73}$$

so, the membership functions of the fuzzy variable $Z = X + Y$ is,

$$\mu_Z(x) = \begin{cases} \varepsilon(\frac{x-(p_1+q_1)}{p_2-p_1+q_2-q_1})^2 & ; \quad p_1+q_1 \le x < p_2+q_2 \\ \varepsilon & ; \quad x = p_2+q_2 \\ \varepsilon(\frac{(p_3+q_3)-x}{p_3-p_2+q_3-q_2})^2 & ; \quad p_2+q_2 \le x < p_3+q_3 \\ 0 & ; \quad otherwise \end{cases} \tag{74}$$

where $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$

Appendix E

Proof—Using the change $z = kx$, we receive $x = z/k$ as well as $x = z/k$ and thus $\varphi(z) = \frac{z}{k}$

so, $\left|\frac{dx}{dz}\right| = \frac{1}{k} = m(z)$.

Therefore,

$$\int_{kp_1}^{X} \eta_1(z)m(z)dz = \int_{kp_1}^{X} \left(\frac{2\left(z-kp_1\right)}{k(p_2-p_1)^2}\right)\left(\frac{1}{k}\right)dz = \left(\frac{x-kp_1}{kp_2-kp_1}\right)^2 \tag{75}$$

$$\int_{kp_3}^{X} \eta_2(z)m(z)dz = \int_{kp_3}^{X} \left(\frac{-2\left(kp_3-z\right)}{k(p_3-p_2)^2}\right)\left(\frac{1}{k}\right)dz = \left(\frac{kp_3-x}{kp_3-kp_2}\right)^2$$

Therefore, at k > 0 is

$$\mu_{kX}(x) = \begin{cases} \varepsilon_1\left(\frac{x-kp_1}{kp_2-kp_1}\right)^2 ; & kp_1 \le x < kp_2 \\ \varepsilon_1 ; & x = kp_2 \\ \varepsilon_1\left(\frac{kp_3-x}{kp_3-kp_2}\right)^2 ; & kp_2 \le x < kp_3 \\ 0 ; & otherwise \end{cases} \tag{76}$$

Also, at k < 0 is

$$\mu_{kX}(x) = \begin{cases} \varepsilon_1\left(\frac{x-kp_3}{kp_2-kp_3}\right)^2 ; & kp_3 \le x < kp_2 \\ \varepsilon_1 ; & x = kp_2 \\ \varepsilon_1\left(\frac{kp_1-x}{kp_1-kp_2}\right)^2 ; & kp_2 \le x < kp_1 \\ 0 ; & otherwise \end{cases} \tag{77}$$

Appendix F

Proof—Because X and Y have a parabolic membership,

$$\mu_x(x) = \begin{cases} \varepsilon_1 L_1(x) & ; \ if\ p_1 \le x < p_2, \\ \varepsilon_1 & ; \ if\ x = p_2. \\ \varepsilon_1 R_1(x) & ; \ if\ p_2 \le x < p_3 \\ 0 & ; \ if\ otherwise \end{cases}$$

And,

$$\mu_Y(y) = \begin{cases} \varepsilon_1 L_1(y) & ; \ if \quad q_1 \le y < q_2, \\ \varepsilon_1 ; & if \quad y = q_2, \\ \varepsilon_1 R_1(y) & ; \ if \quad q_2 \le y < q_3 \\ 0 & ; \ if \quad otherwise \end{cases}$$

Therefore, to find the membership functionsZ = XY at y = $\emptyset_1$(x), z = xy gives

$$x = \frac{\left(p_1q_2 - p_2q_1\right) \pm \sqrt{(p_1q_2 - p_2q_1)^2 + 4\left(q_2-q_1\right)\left(p_2-p_1\right)z}}{2\left(q_2-q_1\right)} = \varphi_1(z) \tag{78}$$

Take,

$$P_1 = \left(p_2-p_1\right)\left(q_2-q_1\right)$$

$$Q_1 = p_1\left(q_2-q_1\right) + q_1\left(p_2-p_1\right)$$

$$R_1 = p_1q_1$$

Hence,

$$\eta_1(z) = \frac{2}{(p_2-p_1)^2} \times \left[\frac{-p_1q_2 - p_2q_1 + 2p_1q_1 + \sqrt{Q_1^2 - 4P_1\left(R_1-z\right)}}{2\left(q_2-q_1\right)}\right] = \frac{1}{p_2-p_1}\left[\frac{-Q_1 + \sqrt{Q_1^2 - 4P_1\left(R_1-z\right)}}{P_1}\right]$$

$$m(z) = \left|\frac{dx}{dz}\right| = \frac{p_2-p_1}{\sqrt{Q_1^2 - 4P_1\left(R_1-z\right)}} \tag{79}$$

therefore,

$$\int_{p_1b_1}^{X} \eta_1(z)m_1(z)dz = \int_{p_1b_1}^{X} \frac{1}{p_2-p_1}\left[\frac{-Q_1 + \sqrt{Q_1^2 - 4P_1\left(R_1-z\right)}}{P_1}\right] \times \frac{p_2-p_1}{\sqrt{Q_1^2 - 4P_1\left(R_1-z\right)}}dz = \int_{p_1b_1}^{X} \frac{1}{P_1}\left[\frac{-Q_1 + \sqrt{Q_1^2 - 4P_1\left(R_1-z\right)}}{\sqrt{Q_1^2 - 4P_1\left(R_1-x\right)}}\right]dz$$

$$= \int_{p_1b_1}^{X} \frac{1}{P_1}\left[\frac{-Q_1}{\sqrt{Q_1^2 - 4P_1\left(R_1-z\right)}} + 1\right]dz = \frac{1}{P_1}\left[\frac{(-Q_1)^2 - Q_1\sqrt{Q_1^2 - 4P_1\left(R_1-x\right)} + 2P_1x - 2P_1R_1}{2P_1}\right]$$

$$= \left[\frac{-Q_1 + \sqrt{Q_1^2 - 4P_1\left(R_1-x\right)}}{2P_1}\right]^2 ; p_1q_1 \le x < p_2q_2 \tag{80}$$

Similarly, by taking.

$$P_2 = (p_3 - p_2)(q_3 - q_2)$$

$$Q_2 = -p_3(q_3 - q_2) - q_3(p_3 - p_2)$$

$$R_2 = p_3 q_3$$

the membership function for the corresponding dispersal functions as.

$$\int_{p_3 q_3}^{X} \eta_1(z) m_1(z) dz = [\frac{-Q_2 + \sqrt{Q_2^2 - 4P_2(R_2 - x)}}{2P_2}]^2 ; p_2 q_2 \le x < p_3 q_3 \tag{81}$$

As a result, the fuzzy variable Z's membership is given by:

$$\mu_{XY}(x) = \begin{cases} \varepsilon(\frac{-Q_1 + \sqrt{Q_1^2 - 4P_1(R_1 - x)}}{2P_1})^2 & p_1 q_1 \le x < p_2 q_2 \\ \varepsilon & x = p_2 q_2 \\ \varepsilon(\frac{-Q_2 + \sqrt{Q_2^2 - 4P_2(R_2 - x)}}{2P_2})^2 & p_2 q_2 \le x < p_3 q_3 \\ 0 & otherwise \end{cases} \tag{82}$$

where, Z = XY.

Appendix G

Proof—let a fuzzy inconstant be $X = [p_1, p_2, p_3, \varepsilon_1]$ whose membership is given in.

$$\mu_x(x) = \begin{cases} \varepsilon_1 L_1(x); & if \quad p_1 \le x < p_{2,} \\ \varepsilon_1; & if \quad x = p_{2,} \\ \varepsilon_1 R_1(x); & if \quad p_2 \le x < p_3 \\ 0 \; ; & if \quad otherwise \end{cases}$$

Then let $z = \frac{1}{X}$ so that $\left|\frac{dx}{dz}\right| = \frac{1}{z^2}$. So, for X^{-1}

$$\int_{X}^{p_1^{-1}} \eta_1(z) m(z) dz = \int_{X}^{p_1^{-1}} (\frac{2}{(p_2 - p_1)^2}\left(\frac{1}{z} - p_1\right))\left(\frac{1}{z^2}\right) dz = (\frac{1 - p_1 x}{x(p_2 - p_1)})^2$$

And,

$$\int_{p_3^{-1}}^{X} \eta_2(z) m(z) dz = \int_{p_3^{-1}}^{X} (\frac{2}{(p_3 - p_2)^2}\left(\frac{1}{z} - p_3\right))\left(\frac{1}{z^2}\right) dx = (\frac{x p_3 - 1}{x(p_3 - p_2)})^2$$

So, constructed on the distribution, fuzzy membership of X^{-1} is

$$\mu_{X^{-1}}(x) = \begin{cases} \varepsilon_1(\frac{x p_3 - 1}{x(p_3 - p_2)})^2 & if \quad p_3^{-1} \le x < p_2^{-1} \\ \varepsilon_1 & if \quad x = p_2^{-1} \\ \varepsilon_1(\frac{1 - p_1 x}{x(p_2 - p_1)})^2 & if \quad p_2^{-1} \le x < p_1^{-1} \\ 0 & otherwise \end{cases} \tag{83}$$

Appendix H

Proof—Deliberate two fuzzy Sigmoid numbers X and Y having membership function as,

$$\mu_X(x) = \begin{cases} \varepsilon_1 L_1(x); & if \quad p_1 \le x < p_{2,} \\ \varepsilon_1; & if \quad x = p_{2,} \\ \varepsilon_1 R_1(x); & if \quad p_2 \le x < p_3 \\ 0; & if \quad otherwise \end{cases}$$

And,

$$\mu_Y(y) = \begin{cases} \varepsilon_1 L_1(y); & if \quad q_1 \le y < q_{2,} \\ \varepsilon_1; & if \quad y = q_{2,} \\ \varepsilon_1 R_1(y); & if \quad q_2 \le y < q_3 \\ 0; & if \quad otherwise \end{cases}$$

We have $Z = X + Y = (p_1 + q_1, p_2 + q_2, p_3 + q_3)$ as the result of combining these fuzzy sigmoid numbers, and if we let $z = x + y$, we obtain $z = x + \varnothing_1(x)$ and $z = x + \varnothing_2(x)$ which means $x = \xi_1(z)$ and $x = \xi_2(z)$ where,

$$x = \xi_1(z) = \frac{(p_2 - p_1) z - (q_1 p_2 - p_1 q_2)}{p_2 + q_2 - p_1 - q_1}$$

hence,

$$m_1(z) = \frac{d}{dz}(\xi_1(z)) = \frac{(p_2 - p_1)}{(p_2 - p_1 + q_2 - q_1)}$$

As in addition,

$$n_1(z) = \frac{d}{dz}(L_1) \text{ at } x = \xi_1(z)$$

$$= \frac{10}{(p_2 - p_1)(\varphi(5) - \varphi(-5))}\left[\varphi\left(x - \frac{p_1 + p_2}{2}\right)\left(\frac{10}{p_2 - p_1}\right)\right]$$

$$\times\left[1 - \varphi\left(x - \frac{p_1 + p_2}{2}\right)\left(\frac{10}{p_2 - p_1}\right)\right]_{x = \xi_1(z)}$$

By solving above equation by part

$$\varphi\left(x-\frac{p_1+p_2}{2}\right)\left(\frac{10}{p_2-p_1}\right)$$

$$=\varphi\left(\frac{(p_2-p_1)z-(q_1p_2-p_1q_2)}{p_2+q_2-p_1-q_1}-\frac{p_1+p_2}{2}\right)\left(\frac{10}{p_2-p_1}\right)$$

$$=\varphi\left(\frac{2(p_2-p_1)z-2(q_1p_2-p_1q_2)-(p_1+p_2)(p_2+q_2-p_1+q_1)}{2(p_2+q_2-p_1-q_1)}-\frac{p_1+p_2}{2}\right)\left(\frac{10}{p_2-p_1}\right)$$

$$=\varphi\left(\frac{2(p_2-p_1)z+(p_1+p_2)(p_1-p_2)+q_2((p_1-p_2)+q_1((p_1-p_2))}{2(p_2+q_2-p_1-q_1)}-\frac{p_1+p_2}{2}\right)\left(\frac{10}{p_2-p_1}\right)$$

$$=\varphi\left(\frac{2z-(p_1+p_2+q_1+q_2)}{2(p_2+q_2-p_1-q_1)}\right)$$

$$=\varphi\left(z-\frac{(p_1+p_2+q_1+q_2)}{2}\right)\left(\frac{10}{(p_2+q_2-p_1-q_1)}\right)$$

Thus,

$$n_1(z)=\frac{10}{(p_2-p_1)(\varphi(5)-\varphi(-5))}\times\varphi\left(z-\frac{(p_1+p_2+q_1+q_2)}{2}\right)\left(\frac{10}{(p_2+q_2-p_1-q_1)}\right)$$

$$\times\left[1-\varphi\left(z-\frac{(p_1+p_2+q_1+q_2)}{2}\right)\left(\frac{10}{(p_2+q_2-p_1-q_1)}\right)\right]_{x=\xi_1(z)}$$

Therefore,

$$L_1(x)=\int_{p_1+q_1}^{X} m_1(z)\eta_1(z)dz$$

$$=\int_{p_1+q_1}^{X}\frac{10(p_2-p_1)}{(p_2-p_1+q_2-q_1)(p_2-p_1)(\varphi(5)-\varphi(-5))}[\varphi(\cdot)(1-\varphi(\cdot))]dz$$

where $\varphi(\cdot)=\varphi\left(z-\frac{(p_1+p_2+q_1+q_2)}{2}\right)\left(\frac{10}{(p_2+q_2-p_1-q_1)}\right)$

$$=\frac{1}{(\varphi(5)-\varphi(-5))}\left[\varphi\left(\left\{z-\frac{p_1+p_2+q_1+q_2}{2}\right\}\frac{10}{(p_2+q_2-p_1-q_1)}\right)\right]_{p_1+q_1}^{X}$$

$$=\frac{1}{(\varphi(5)-\psi(-5))}\left[\varphi\left(\left\{x-\frac{p_1+p_2+q_1+q_2}{2}\right\}\frac{10}{(p_2+q_2-p_1-q_1)}\right)\right.$$

$$\left.-\varphi\left(\left\{p_1+q_1-\frac{p_1+p_2+q_1+q_2}{2}\right\}\frac{10}{(p_2+q_2-p_1-q_1)}\right)\right]$$

$$=\frac{1}{(\varphi(5)-\varphi(-5))}\left[\varphi\left(\left\{x-\frac{p_1+p_2+q_1+q_2}{2}\right\}\frac{10}{(p_2+q_2-p_1-q_1)}\right)-\varphi(-5)\right]$$

So, the distribution function of left side fuzzy number Z = X + Y is;

$$L_1(z)=\frac{\varphi\left(\left\{z-\frac{p_1+p_2+q_1+q_2}{2}\right\}\frac{10}{(p_2+q_2-p_1-q_1)}\right)-\varphi(-5)}{\varphi(5)-\varphi(-5)}$$

Likewise, if $y=\varphi_2(x)$, then x = x + y becomes x = ξ_2(z), where

$$\xi_2(z)=\frac{(p_3-p_2)z-(q_2p_3-p_2q_3)}{(p_3-p_2)+(q_3-q_2)}$$

Thus, $m_2(z)=\frac{d(\xi_2(z))}{dx}=\frac{p_3-p_2}{(p_3-p_2)+(q_3-q_2)}$
and,

$$\eta_2(x) = \frac{d}{dx}\left(R_1(x)\right) at x = \xi_2(z)$$

$$= \frac{-10}{\left(p_3 - p_2\right)(\varphi(5) - \varphi(-5))}\varphi \times \left(\left\{x - \frac{p_2 + p_3}{2}\right\}\frac{10}{p_3 - p_2}\right) \times \left(1 - \varphi\left(\left\{x - \frac{p_2 + p_3}{2}\right\}\frac{10}{p_3 - p_2}\right)\right)$$

At $x = \xi_2(z) = \frac{(p_3-p_2)z-(q_2p_3-p_2q_3)}{(p_3-p_2)+(q_3-q_2)}$, we have

By solving above equation by part

$$\varphi\left(\left\{x - \frac{p_2 + p_3}{2}\right\}\frac{10}{p_3 - p_2}\right) = \varphi\left(\left\{\frac{\left(p_3 - p_2\right)x - \left(q_2p_3 - p_2q_3\right)}{\left(p_3 - p_2\right) + \left(q_3 - q_2\right)} - \frac{p_2 + p_3}{2}\right\} \times \frac{10}{p_3 - p_2}\right)$$

$$= \varphi\left(\frac{2\left(p_3 - p_2\right)x - 2\left(q_2p_3 - p_2q_3\right) - \left(p_2 + p_3\right)\left(p_3 + q_3 - p_2 - q_2\right)}{2\left(p_3 + q_3 - p_2 - q_2\right)} \times \frac{10}{p_3 - p_2}\right)$$

$$= \varphi\left(\frac{2\left(p_3 - p_2\right)x + \left(p_2 + p_3\right)\left(p_2 - p_3\right) + q_2\left(p_2 - p_3\right) + q_3\left(p_2 - p_3\right)}{2\left(p_3 + q_3 - p_2 - q_2\right)} \times \frac{10}{p_3 - p_2}\right)$$

$$= \varphi\left(\left\{\frac{2x - \left(p_2 + p_3 + q_2 + q_3\right)}{2\left(p_3 + q_3 - p_2 - q_2\right)}\right\} \times 10\right)$$

$$= \varphi\left(\left\{z - \frac{p_3 + p_2 + q_2 + q_3}{2}\right\}\frac{10}{p_3 + q_3 - p_2 - q_2}\right)$$

$$\eta_2(z) = \frac{10}{\left(p_3 - p_2\right)(\varphi(5) - \varphi(-5))}\varphi\left(\left\{z - \frac{p_3 + p_2 + q_2 + q_3}{2}\right\}\frac{10}{p_3 + q_3 - p_2 - q_2}\right) \times\left[1 - \varphi\left(\left\{z - \frac{p_3 + p_2 + q_2 + q_3}{2}\right\}\frac{10}{p_3 + q_3 - p_2 - q_2}\right)\right]$$

Therefore,

$$R_1(x) = \int_x^{p_3+q_3} \eta_2(z)m_2(z)dz$$

$$= \int_x^{p_3+q_3} \frac{10\left(p_3 - p_2\right)}{\left(p_3 - p_2 + q_3 - q_2\right)\left(p_3 - p_2\right)(\psi(5) - \psi(-5))}[\varphi(\cdot)(1 - \varphi(\cdot))]dz$$

where

$$\varphi(\cdot) = \varphi\left(\left\{z - \frac{p_3 + p_2 + q_2 + q_3}{2}\right\}\frac{10}{p_3 + q_3 - p_2 - q_2}\right)$$

$$= \frac{1}{(\varphi(5) - \varphi(-5))}\left[\varphi\left(\left(z - \frac{p_3 + p_2 + q_2 + q_3}{2}\right)\frac{10}{p_3 + q_3 - p_2 - q_2}\right)\right]_x^{p_3+q_3}$$

$$= \frac{1}{(\varphi(5) - \varphi(-5))}\left[\varphi\left(\left(p_3 + q_3 - \frac{p_3 + p_2 + q_2 + q_3}{2}\right)\frac{10}{p_3 + q_3 - p_2 - q_2}\right) - \varphi\left(\left(x - \frac{p_3 + p_2 + q_2 + q_3}{2}\right)\frac{10}{p_3 + q_3 - p_2 - q_2}\right)\right]$$

$$= \frac{\varphi(5) - \varphi\left(\left\{x - \frac{p_3+p_2+q_2+q_3}{2}\right\}\frac{10}{p_3+q_3-p_2-q_2}\right)}{\varphi(5) - \varphi(-5)}$$

As a result, the right-side fuzzy number (Z = X + Y)'s distribution function is;

Similarly, the non-complementary function $R_1(z)$, gives.

$$R_1(z) = \frac{\varphi(5) - \varphi\left(\left\{z - \frac{p_3+p_2+q_2+q_3}{2}\right\}\frac{10}{p_3+q_3-p_2-q_2}\right)}{\varphi(5) - \varphi(-5)}$$

As a result, the fuzzy variable Z = X + Y membership is given by

$$L_1(x) = \int_{kp_1}^{x} \eta_1(z)m(z)dz$$

$$= \int_{kp_1}^{x}\left[\frac{10}{k(p_2-p_1)(\varphi(5)-\varphi(-5))}\left(\varphi\left(\left\{\frac{z}{k} - \frac{p_1+p_2}{2}\right\}\frac{10}{p_2-p_1}\right)\right) \times\left(1-\varphi\left(\left\{\frac{z}{k} - \frac{p_1+p_2}{2}\right\}\frac{10}{p_2-p_1}\right)\right)\right]dz$$

$$= \frac{\varphi\left(\left\{x - \frac{kp_1+kp_2}{2}\right\}\frac{10}{(kp_2-kp_1)}\right) - \varphi(-5)}{\varphi(5)-\varphi(-5)} \tag{85}$$

$$\mu_z(Z) = \begin{cases} \varepsilon\frac{\varphi\left[\left(z - \frac{p_1+p_2+q_1+q_2}{2}\right)\left(\frac{10}{p_2+q_2-p_1-q_1}\right)\right] - \varphi(-5)}{\varphi(5)-\varphi(-5)} & ;\ if\quad p_1+q_1 \le z < p_2+q_2 \\ \varepsilon & ;\ if\quad z = p_2 + q_2 \\ \varepsilon\frac{\varphi(5) - \varphi\left[\left(z - \frac{p_2+p_3+q_2+q_3}{2}\right)\left(\frac{10}{p_3+q_3-p_2-q_2}\right)\right]}{\varphi(5)-\varphi(-5)} & ;\ if\quad p_2+q_2 \le z < p_3+q_3 \\ 0 & ;\quad otherwise \end{cases} \tag{84}$$

Appendix I

Proof—By the use of transformation given by z = kx, we get $x = \frac{z}{k}$, which can be written as ξ(z)

so, $\left|\frac{dx}{dz}\right| = \frac{1}{k} = m(z)$. Then, at $x = \xi(z)$

$$\varphi(z) = \varphi\left(\left\{\frac{z}{k} - \frac{p_1+p_2}{2}\right\}\frac{10}{(p_2-p_1)}\right)$$

In addition, we have.

$$\eta_1(z) = \frac{d}{dx}(L_1) \text{ at } x = \xi_1(z) = \frac{z}{k}$$

$$\eta_2(z) = \frac{d}{dx}(R_1(x)) \text{ at } x = \xi(z) = \frac{z}{k}$$

$$= \frac{10}{(p_3-p_2)(\varphi(5)-\varphi(-5))}\varphi\left(\left\{\frac{z}{k} - \frac{p_2+p_3}{2}\right\}\frac{10}{p_3-p_2}\right) \times\left(1-\varphi\left(\left\{\frac{z}{k} - \frac{p_2+p_3}{2}\right\}\frac{10}{p_3-p_2}\right)\right)$$

And therefore,

$$R_1(x) = \int_{X}^{kp_3} \eta_2(z)m(z)dx$$

$$= \int_{X}^{kp_3}\left[\frac{10}{(p_3-p_2)(\varphi(5)-\varphi(-5))}\times\left(\varphi\left(\left\{\frac{z}{k} - \frac{p_2+p_3}{2}\right\}\frac{10}{p_3-p_2}\right)\right)\times\left(1-\varphi\left(\left\{\frac{z}{k} - \frac{p_2+p_3}{2}\right\}\frac{10}{p_3-p_2}\right)\right)\right]dz$$

$$= \frac{10}{(p_2-p_1)(\varphi(5)-\varphi(-5))}\times\left(\varphi\left(\left\{\frac{z}{k} - \frac{p_1+p_2}{2}\right\}\frac{10}{p_2-p_1}\right)\right) \times\left(1-\varphi\left(\left\{\frac{z}{k} - \frac{p_1+p_2}{2}\right\}\frac{10}{p_2-p_1}\right)\right)$$

Hence,

$$= \frac{\varphi(5) - \varphi\left(\left\{x - \frac{kp_2+kp_3}{2}\right\}\frac{10}{(kp_3-kp_2)}\right)}{\varphi(5)-\psi(-5)}$$

The membership for the fuzzy number kX at k > 0 is:

$$\mu_{kX}(x)=\begin{cases}\varepsilon_1\frac{\varphi\left(\left\{x-\frac{kp_1+kp_2}{2}\right\}\frac{10}{(kp_2-kp_1)}\right)-\varphi(-5)}{\varphi(5)-\varphi(-5)}; & kp_1\le x<kp_2\\ \varepsilon_1; & x=kp_2\\ \varepsilon_1\frac{\varphi(5)-\varphi\left(\left\{x-\frac{kp_2+kp_3}{2}\right\}\frac{10}{(kp_3-kp_2)}\right)}{\varphi(5)-\varphi(-5)}; & kp_2\le x<kp_3.\end{cases}$$

Likewise, the membership for the fuzzy number kX at k < 0, is:

$$\mu_{kX}(x)=\begin{cases}\varepsilon_1\frac{\varphi\left(\left\{x-\frac{kp_2+kp_3}{2}\right\}\frac{10}{(kp_2-kp_3)}\right)-\varphi(-5)}{\varphi(5)-\varphi(-5)}; & kp_3\le x<kp_2\\ \varepsilon_1; & x=kp_2\\ \varepsilon_1\frac{\varphi(5)-\varphi\left(\left\{x-\frac{kp_1+kp_2}{2}\right\}\frac{10}{(kp_1-kp_2)}\right)}{\varphi(5)-\varphi(-5)}; & kp_2\le x<kp_1\end{cases}$$

Appendix J

Proof—The fuzzy number $Z = XY$, associate $L_1(x)$ with $L_1(y)$ and $R_1(x)$ with $R_2(y)$ and get $y = \emptyset_1(x)$ and $y = \emptyset_2(x)$,

Where, $\emptyset_1(x)=\frac{q_1+q_2}{2}+\frac{q_2-q_1}{a_2-a_1}\left(x-\frac{a_1+a_2}{2}\right)$; $\emptyset_2(x)=\frac{q_2+q_3}{2}+\frac{q_3-q_2}{a_3-a_2}\left(x-\frac{a_2+a_3}{2}\right)$.

Therefore, at $y = \emptyset_1(x)$, $z = xy$ becomes

$$x=\frac{p_1q_2-p_2q_1\pm\sqrt{(p_1q_2-p_2q_1)^2+4(p_2-p_1)(q_2-q_1)x}}{2(q_2-q_1)}=\xi_1(z)$$

Therefore,

$$m_1(z)=|\frac{dx}{dz}|_{x=\xi_1(z)}$$

$$=\frac{4(p_2-p_1)(q_2-q_1)}{4(q_2-q_1)\sqrt{(p_1q_2-p_2q_1)^2+4(p_2-p_1)(q_2-q_1)z}}$$

$$=\frac{p_2-p_1}{\sqrt{(p_1q_2-p_2q_1)^2+4(p_2-p_1)(q_2-q_1)z}}=\frac{p_2-p_1}{\sqrt{Q_1^2-4P_1(R_1-z)}}$$

where $P_1=(p_2-p_1)(q_2-q_1)$; $Q_1=p_1(q_2-q_1)+$;$R_1=p_1q_1$.and in addition,

$$\eta_1(z)=\frac{d}{dx}(L_1)\text{ at, }x=\xi_1(z)$$

$$=\frac{10}{(\varphi(5)-\varphi(-5))dx}\left(\varphi\left(\left\{x-\frac{p_1+p_2}{2}\right\}\frac{10}{(p_2-p_1)}\right)\right)$$

$$=\frac{10}{(p_2-p_1)(\varphi(5)-\varphi(-5))}\varphi\left(\left\{x-\frac{p_1+p_2}{2}\right\}\frac{10}{(p_2-p_1)}\right)\times\left(1-\varphi\left(\left\{x-\frac{p_1+p_2}{2}\right\}\frac{10}{(p_2-p_1)}\right)\right)$$

At,

$$X=\xi_1(z)=\frac{p_1q_2-p_2q_1\pm\sqrt{(p_1q_2-p_2q_1)^2+4(p_2-p_1)(q_2-q_1)z}}{2(q_2-q_1)}$$

$$\varphi\left(\left\{x-\frac{p_1+p_2}{2}\right\}\frac{10}{(p_2-p_1)}\right)=\varphi\left(\left\{\frac{p_1q_2-p_2q_1\pm\sqrt{(p_1q_2-p_2q_1)^2+4(p_2-p_1)(q_2-q_1)z}p_1+p_2}{2(q_2-q_1)2}\right\}\times\frac{10}{p_2-p_1}\right)$$

$$=\varphi\left(\left\{\frac{p_1q_2-p_2q_1\pm\sqrt{Q_1^2-4P_1(R_1-z)}p_1+p_2}{2(q_2-q_1)2}\right\}\frac{10}{p_2-p_1}\right)$$

$$=\varphi\left(\left\{\frac{p_1q_1-p_2q_2\pm\sqrt{Q_1^2-4P_1(R_1-z)}}{2(q_2-q_1)}\right\}\frac{10}{p_2-p_1}\right)$$

Hence

$$\eta_1(z)=\frac{10}{(p_2-p_1)(\psi(5)-\psi(-5))}\varphi\left(\left\{\frac{p_1q_1-p_2q_2\pm\sqrt{Q_1^2-4P_1(R_1-z)}}{2(q_2-q_1)}\right\}\frac{10}{p_2-p_1}\right)$$

$$\times\left[1-\varphi\left(\left\{\frac{p_1q_1-p_2q_2\pm\sqrt{Q_1^2-4P_1\left(R_1-z\right)}}{2\left(q_2-q_1\right)}\right\}\frac{10}{p_2-p_1}\right)\right]$$

Therefore,

$$L_1(x)=\int_{p_1q_1}^{X}m_1(z)\eta_1(z)dz=\int_{p_1q_1}^{X}\frac{10}{(\varphi(5)-\psi(-5))\sqrt{Q_1^2-4P_1\left(R_1-z\right)}}\varphi(\cdot)(1-\varphi(\cdot))dz$$

where $\varphi(\cdot)=\varphi\left(\left\{\frac{p_1q_1-p_2q_2\pm\sqrt{Q_1^2-4P_1\left(R_1-z\right)}}{2\left(q_2-q_1\right)}\right\}\frac{10}{p_2-p_1}\right)$

$$=\frac{1}{(\varphi(5)-\varphi(-5))}\left[\varphi\left(\left\{\frac{p_1q_1-p_2q_2\pm\sqrt{Q_1^2-4P_1\left(R_1-z\right)}}{2\left(q_2-q_1\right)}\right\}\frac{10}{p_2-p_1}\right)\right]_{p_1b_1}^{X}$$

$$=\frac{1}{(\varphi(5)-\varphi(-5))}\left[\varphi\left(\left\{\frac{p_1q_1-p_2q_2\pm\sqrt{(Q_1)^2-4P_1\left(R_1-x\right)}}{2\left(q_2-q_1\right)\left(p_2-p_1\right)}\right\}10\right)-\varphi\left(\left\{\frac{p_1q_1-p_2q_2\pm\sqrt{Q_1^2-4P_1\left(Q_1-p_1q_1\right)}}{2\left(q_2-b_1\right)\left(p_2-p_1\right)}\right\}\right.\right.$$

$$=\frac{\varphi\left(\left\{\frac{p_1q_1-p_2q_2\pm\sqrt{Q_1^2-4P_1\left(R_1-x\right)}}{2\left(q_2-q_1\right)\left(p_2-p_1\right)}\right\}10\right)-\varphi(-5)}{\varphi(5)-\varphi(-5)}$$

$=\frac{\varphi\left(\tau_1\right)-\varphi(-5)}{\varphi(5)-\varphi(-5)}$, where, $\tau_1=10\left(\frac{-Q_1+\sqrt{Q_1^2-4P_1\left(R_1-x\right)}-P_1}{2P_1}\right)$.

By taking, $P_2=\left(p_3-p_2\right)\left(q_3-q_2\right)$; $Q_2=-p_3\left(q_3-q_2\right)-q_3\left(p_3-p_2\right)$; $R_2=p_3q_3$,

We're having the accurate membership of $Z=XY$ as:

$R_1(x)=\int^{p_3q_3}\eta_2(z)m_2(z)dz=\frac{\varphi(5)-\varphi\left(\tau_2\right)}{\varphi(5)-\varphi(-5)}$, where $\tau_2=10\left(\frac{-Q_2-\sqrt{Q_2^2-4P_2\left(R_2-x\right)}+P_2}{2P_2}\right)$ consequently, the membership of the fuzzy number $Z=XY$ is:

$$\mu_{XY}(z)=\begin{cases}\varepsilon\frac{\varphi\left(\tau_1\right)-\varphi(-5)}{\varphi(5)-\varphi(-5)}; & p_1q_1\le z<p_2q_2\\ \varepsilon; & z=p_2q_2\\ \varepsilon\frac{\varphi(5)-\varphi\left(\tau_2\right)}{\varphi(5)-\varphi(-5)}; & p_2q_2\le z<p_3q_3\end{cases}\tag{86}$$

where $\tau_1=10\left(\frac{-Q_1+\sqrt{Q_1^2-4P_1\left(R_1-z\right)}-P_1}{2P_1}\right)$ and $\tau_2=10\left(\frac{-Q_2-\sqrt{Q_2^2-4P_2\left(R_2-z\right)}+P_2}{2P_2}\right)$

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