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Exponential Diophantine Equations Involving Opposite Parity Prime

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Abstract

Many researchers have been devoted to finding the solutions (x,y,z) in the set of nonnegative integers, of Diophantine equations of the type $p^x+q^y=z^2$, where the values and q are fixed. In this article, we demonstrate that few singular Exponential Diophantine equations

$$E_1: 2^x+7^y=z^2$$
$$E_2: 2^x+41^y=z^2,$$
$$E_3: 2^x+43^y=z^2,$$
$$E_4: 2^x+23^y=z^2$$
$$E_5: 2^x+31^y=z^2$$

has only a finite number of solutions in $N\cup\{0\}$. The solution sets (x,y,z) of E_1,E_2,E_3,E_4 and E_5 are $\{$(1,1,3),(3,0,3),(5,2,9)$\}$ $\{$(3,0,3),(3,1,7),(7,1,13)$\}$,$\{$(3,0,3)$\}$,$\{$(3,0,3),(1,1,5)$\}$ and respectively.

Keywords

Exponential Diophantine equation, Congruence, Integral points, Catalan's Conjecture, Co prime.

AMS Subject Classification

11D61, 11D72.

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Contents

1. Introduction

Number Theory is a division of pure mathematics faithful primarily to the revision of integers. The Diophantine investigation deals with an assortment of techniques for solving Diophantine equations in multivariable's and multi degrees. A Diophantine equation is a polynomial equation that takes only integer values. There are various forms of Diophantine equations studied by different mathematicians $[1-4,7,8]$ in the last couple of decades. If a Diophantine equation has variables happening as exponents, it is an exponential Diophantine equation. For example the Ramanujan $-$ Nagell equation $2^x-7=x^2$ and the equation of the Fermat $-$ Catalan conjecture $a^m+b^n=c^k$.

For related papers, we list them as follows. In 2007, Acu [1] proved that (3,0,3) and (2,1,3) are only two solutions (x,y,z) for the Diophantine equation $2^x+5^y=z^2$ where x,y, and z are non-negative integers. In 2011, Suvarnamani, Singta and Chotchaisthit [5] proved that the two Diophantine equations $4^x+7^y=z^2$ and $4^x+11^y=z^2$ have no non - negative integer solution. In 2012, Chotchaisthit [3] found all non - negative integer solutions for the Diophantine equation of type $4^x+p^y=z^2$ where p is a prime number.

2. Preliminaries

In this section, we use the factorizable technique and Catalan's Conjecture to establish the four lemmas.

Proposition 2.1 ([4]). *(The Catalan's conjecture) (3,2,2,3) is* *a unique solution (a,b,x,y) for the Diophantine equation $a^x - b^y = 1$ where a,b,x and y are integers with* $\min\{a,b,x,y\} \geq 2$.

Lemma 2.2. *The Diophantine equation $2^x + 1 = z^2$ has only a unique solution (3,3) in $N \cup \{0\}$.*

Proof. Suppose that there are non - negative integers x and z such that $2^x + 1 = z^2$. If $x = 0$, $z^2 = 2$, which is impossible. Therefore, $x \geq 1$. Thus, $z^2 = 2^x + 1 \geq 2^1 + 1 = 3$. Then $z \geq 4$ Now, we think the equation $z^2 - 2^x = 1$. By Proposition 2.1, we have only the solutions, $x = 3$ and $z = 3$. Hence, the equation $2^x + 1 = z^2$ has a unique solution (3,3) in $N \cup \{0\}$. □

Another Proof: The equation can be written as $z^2 - 1 = 43^y$. Then $(z+1)(z-1) = 43^y$ implies that $(z+1)(z-1) = 43^{y-u} \cdot 43^u$ which is equivalent to $z+1 = 43^{y-u}$ and $z-1 = 43^u$. Thus it follows that $2 = 43^{y-u} - 43^u$ this implies that $2.1 = 43^u\left(43^{y-2u} - 1\right)$. Thus $u = 0$ is the only possible. Therefore $2 = 43^y - 1$ implies that $43^y = 3$. This is a contradiction. Therefore there is no non - negative integer solution exists.

3. Main Results

Theorem 3.1. *Prove that the number of triplets (x,y,z) of non-negative integers such that $2^x + 7^y = z^2$ are three.*

Proof. Let $x, y,$ and z be non-negative integers such that $2^x + 7^y = z^2$. ByLemma 2.2, we have $x \geq 1$. Now, we divide the number y into two cases.
Case (i): If $y = 0$. By Lemma 2.2 , we have $x = 3$ and $z = 3$. Therefore the solution to the Diophantine equation E_1 is (3,0,3).
Case (ii): If $y = 1$, then z will be odd. This implies $z^2 \equiv 1(\bmod 4)$. So $2^x \equiv 2(\bmod 4)$. It is only possible that the case is $x = 1$. From the Diophantine equation E_1, we obtain $z = 3$. Finally, we conclude that the solution to this particular case is (1,1,3).
Case (iii): Suppose $y > 1$, Now z will be odd. Then, $z^2 \equiv 1(\bmod 4)$. This implies that $7^y \equiv 1(\bmod 4)$. Thus, y is even. Let $y = 2k$, where $k \in N$. Then $z^2 - 7^{2k} = 2^x$ implies that $\left(z+7^k\right)\left(z-7^k\right) = 2^x$ which equivalent is to $\left(z+7^k\right) = 2^{x-u}$ and $\left(z-7^k\right) = 2^u$. Thus, it follows that $2\left(7^k\right) = 2^{x-u} - 2^u = 2^u\left(2^{x-2u} - 1\right)$ which implies $2 = 2^u$ and $2^{x-2u} - 1 = 7^k$, its only possible $u = 1$. It gives $7^k = 2^{x-2u} - 1$. As $k \in N$, then $x - 2u \geq 3$. By Proportion 2.1, we have $k = 1$. Therefore, the only possible $x - 2u = 3$. Hence $x = 5$ as $u = 1$, therefore $z = 9$ and $y = 2$. We conclude that which only a suitable solution is (5,2,9). □

Theorem 3.2. *The number of non - negative integral solutions to the Diophantine equation $E_3 : 2^x + 43^y = z^2$ is only one solution.*

Proof. Let x,y and z be non $-$ negative integers such that $2^x + 43^y = z^2$. Suppose $y = 0$ then the equation becomes $2^x + 1 = z^2$. By Lemma 2.2, (3,3) is the unique solution. Thus when $y = 0, (3,0,3)$ is the non $-$ negative integral solution for the Diophantine equation $2^x + 43^y = z^2\cdot$ Now, we divide x into three cases.
Case (i): If $x = 0$. By Lemma 2.4, there is no non - negative integral solution exist for the equation $E_3 : 2^x + 43^y = z^2$.
Case (ii): If $x = 1$. Then $E_3 : 2^x + 43^y = z^2$ becomes $43^y = z^2 - 2$. Then y must be odd. Take $y = 2k+1$. Then the equation becomes $43^{2k+1} = z^2 - 2$ implies that $43\left(43^{2k}\right) = (z+\sqrt{2})(z-\sqrt{2})$. This is not possible. Therefore, when $x = 1$, there is no solution exists for E_3.
Case (iii): If $x > 1$. In this case $2^x \equiv 0(\bmod 4)$. Also $z^2 \equiv 1(\bmod 4)$. This implies that $43^y \equiv 1(\bmod 4)$. Therefore y must be even. Take $y = 2k, k = 1,2,\ldots$ Now the equation becomes $z^2 - 43^{2k} = 2^x$ implies that $\left(z+43^k\right)\left(z-43^k\right) = 2^{x-u}\cdot 2^u$ implies $\left(z+43^k\right) = 2^{x-u}$ and $\left(z-43^k\right) = 2^u$. It follows that $2\left(43^k\right) = 2^u\left(2^{x-2u} - 1\right)\cdot u = 1$ is the only possible value. Thus $43^k = 2^{x-2} - 1$. Since $k > 0, 2^{x-2} \geq 44$ implies that $x \geq 8$ By Proposition 2.1, k must be equal to 1. Therefore $2^{x-2} = 44$. This is a contradiction to x is a non - negative integer. Thus we conclude that (3,0,3) is a unique solution for the Diophantine equation E_3. Pictorial representation of the equation $2^x + 43^y = z^2$ □

Theorem 3.3. *The number of non negative integral solutions to the Diophantine equation $E_4 : 2^x + 23^y = z^2$ is only two.*

Proof. We will divide the number x into three cases.
Case (i): If $x = 0$. Then, $1 + 23^y = z^2$. If $y = 0$, then $z^2 = 2$. It is not possible. Therefore $y \geq 1.z^2 = 1 + 23^y \geq 24 \Rightarrow z \geq 5$. By Proposition 2.1, y must be equal to 1. $z^2 = 24$ Since z is non - negative integer, it is impossible. Therefore, when $x = 0$, there is no such solution exists for $2^x + 23^y = z^2$.
Case (ii): If $x = 1$. Then $2 + 23^y = z^2$. Since z is odd $z^2 \equiv 1(\bmod 4)$. This implies $23^y \equiv 3(\bmod 4)$. Therefore y must be odd. Take $y = 2k + 1.2 + 23^{2k+1} = z^2$. $z^2 - 2 = 23^{2k+1}\cdot z^2 - 2 = 23\left(23^{2k}\right)$. When k $= 0, z^2 - 2 = 23 \Rightarrow z^2 = 25 \Rightarrow z = 5$ $k = 0 \Rightarrow y = 1$. Therefore (1,1,5) is the solution for $2^x + 23^y = z^2$. When $k \neq 0$ $z^2 - 2 = 23\left(23^{2k}\right)$ is not possible. Therefore, when $x = 1, (1,1,5)$ is the only solution for $2^x + 23^y = z^2$.
Case (iii): If $x > 1$. Since $> 1, 2^x \equiv 0(\bmod 4)$. Since $z^2 \equiv 1(\bmod 4), 23^y \equiv 1(\bmod 4)$ This gives y must be even $y = 2k$.

$$2^x + 23^{2k} = z^2$$
$$z^2 - 23^{2k} = 2^x$$
$$\left(z+23^k\right)\left(z-23^k\right) = 2^{x-u}\cdot 2^u$$
$$\Rightarrow 2\left(23^k\right) = 2^{x-u} - 2^u$$
$$\Rightarrow 2\left(23^k\right) = 2^u\left(2^{x-2u} - 1\right)$$

Here $u = 1$ is the only possible value. Then $23^k = 2^{x-2} - 1.2^{x-2} - 23^k = 1$. When $k = 0.2^{x-2} = 1 + 23^0.2^{x-2} \Rightarrow x - 2 = 1 \Rightarrow x = 3.k = 0 \Rightarrow y = 0$. Therefore, $2^3 + 23^0 = z^2 \Rightarrow z^2 = 9 \Rightarrow z = 3$. Therefore (3,0,3) is the solution. Assume $k \geq 1$. Then $2^{x-2} = 1 + 23^k \geq 24$. Therefore by Catalan's conjecture, k must be equal to $1.2^{x-2} = 24$ which is not possible.

Therefore, If $x>1,(3,0,3)$ is the only solutions for E_4. Thus (3,0,3) and (1,1,5) are the only solutions for E_4 Pictorial representation of the equation $2^x+23^y=z^2$: □

Theorem 3.4. *The number of non negative integral solutions to the Diophantine equation $E_5: 2^x+31^y=z^2$ is only two.*

Proof. We divide the value of x into three cases.
Case (i): If $x=0$. Then $1+31^y=z^2$ has no non $-$ negative integer solution. For, $y=0, z^2=2$ is impossible. Therefore, $y \geq 1$ ie., $z^2 \geq 32 \Rightarrow z \geq 5$. By Catalan's conjecture, y must be equal to 1. $y=1 \Rightarrow z^2=32$. This is also impossible. Therefore, when $x=0$, there exists no non $-$ negative solution for $2^x+31^y=z^2$.
Case (ii): If, $x=1$. Then, $2+31^y=z^2$. Since $z^2 \equiv 1(\bmod 4), 31^y \equiv 3(\bmod 4) \Rightarrow y$ is odd. $\Rightarrow y=2k+1 . z^2=2+31^{2k+1}$ $\Rightarrow z^2-2=31\left(31^{2k}\right)$. When $k=0, z^2=33$ which is absurd. When, $k \neq 0 . z^2-2=31\left(31^{2k}\right)$ is not possible. Thus when $x=1$, we cannot find any solution for $2^x+31^y=z^2$.
Case (iii): If, $x>1$. When $x>1, 2^x \equiv 0(\bmod 4)$. We know that, $z^2 \equiv 1(\bmod 4)$. This, gives $31^y \equiv 1(\bmod 4)$ y must be even integer. $y=2k$. Therefore,
mu $z^2-31^{2k}=2^x\left(z+31^k\right)\left(z-31^k\right)=2^{x-u} \cdot 2^u \Rightarrow 2\left(31^k\right)=2^u\left(2^{x-2u}-1\right) 2=2^u$, and $31^k=2^{x-2u}-1$. Here $u=1$ is only possible. $u=1, 31^k=2^{x-2}-1.2^{x-2}-31^k=1$ when $k=0, 2^{x-2}=2 \Rightarrow x=3$.

Therefore (3,0,3) is the solution. Therefore let us assume that $k \geq 1.2^{x-2} \geq 32 \Rightarrow x \geq 7$. Therefore by the Catalan's conjecture, k must be equal to 1. Therefore $2^{x-2}-31=1.2^{x-2}=32=2^5 \Rightarrow x=7$. $k=1 \Rightarrow y=2\ 2^x+31^y=z^2 \Rightarrow$ $2^7+31^2=z^2 \Rightarrow 128+961=z^2 \Rightarrow 1089=z^2 \Rightarrow z=33$. Therefore (7,2,33) is the solution. Thus (3,0,3) and (7,2,33) are the only solutions for $2^x+31^y=z^2$. Pictorial representation of the equation $2^x+31^y=z^2$: □

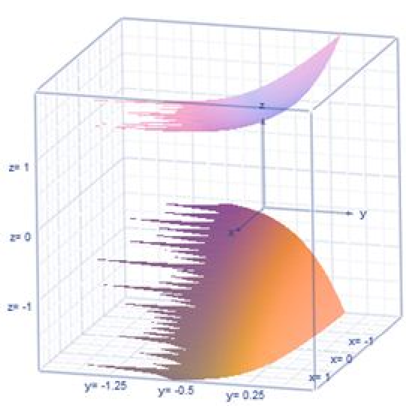

Figure 1

4. Conclusion

We note in our results that 2 is an even prime number and $7-2=5$. Let p be an odd prime number. We may ask for the set of all solutions (x,y,z) for the Diophantine equation $p^x+(p+n)^y=z^2$ where x, y, and z are non-negative integers.

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