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ORIGINAL RESEARCH PAPER

On common fixed points of non-Lipschitzian semigroups in a hyperbolic metric space endowed with a graph

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Abstract

In this paper, the existence theorem of a common fixed point for a semigroup of contraction mappings defined on a metric space endowed with a graph **G** has been proved. Also, the existence theorem of a common fixed point for a semigroup of **G**-nearly asymptotically nonexpansive mappings in a hyperbolic metric space endowed with a graph has been proved.

Keywords Hyperbolic metric space · **G**-nearly asymptotically nonexpansive mapping · Fixed point theorems · Semigroup of mappings

Mathematics Subject Classification 47H10

1 Introduction

A point $p \in \chi$ is said to be a fixed point of a self-mapping S on χ whenever $Sp = p$ and the set of all fixed points of S is denoted by $F(S)$. In 1922, Banach proved the first fundamental theorem of fixed point theory, which is known as the Banach contraction principle. After the Banach contraction principle, several generalizations

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came into the picture, see [4–7, 9, 11, 13, 20]. The breakthrough generalization was given by Ran and Reurings [17] and Nieto and López [16]. They generalized the Banach contraction principle to partially ordered metric spaces. Ran and Reurings [17] applied their results to solve matrix equations, while Nieto and López [16] applied their results to solve differential equations.

A partial order relation should satisfy reflexivity, anti-symmetricity, and transitivity, whereas a graph does not need any condition. So, the graph is a weaker notion than the partially ordered relation. Recently, in 2007, Jachymski [12] gave some generalizations of the Banach contraction principle for mappings defined on a metric space endowed with a graph. In 2016, Alfuraidan and Shukri [2] proved an analog to Browder and Göhde fixed point theorem for **G**-nonexpansive mappings.

Due to their applications in mathematics and outside mathematics, for instance, in dynamical systems, the existence and convergence theorems concerning fixed points of a semigroup of mappings have been intensely studied for more than half a century. Recently, in 2018, Bachar et al. [3] gave the existence theorem of a common fixed point of a monotone contraction semigroup in metric spaces and the existence theorem of a common fixed point of a monotone nonexpansive semigroup in hyperbolic metric spaces.

One of the most important non-Lipschitzian type mappings, namely nearly asymptotically nonexpansive mapping, which is a generalization of asymptotically nonexpansive mapping, was first introduced by Sahu [19] in 2005. As recently as 2019, Aggarwal et al. [1] proved the existence theorem of a fixed point and some convergence results for monotone nearly asymptotically nonexpansive mappings in partially ordered hyperbolic metric spaces.

In this work, we study the existence of a common fixed point for a semigroup of contraction mappings in a complete metric space endowed with a graph. Further, we give the existence theorem of a common fixed point for a semigroup of **G**-nearly asymptotically nonexpansive mappings in a complete uniformly convex hyperbolic metric space endowed with a graph. Hence, our results are generalizations of those given by Aggarwal et al. [1] and Bachar et al. [3].

2 Preliminaries

To make our paper self-contained, we recall some definitions and present results that will be needed in Sect. 3.

Let (χ, ρ) be a metric space. A graph $\mathbf{G}$ is an ordered pair $(V(\mathbf{G}), E(\mathbf{G}))$, where $V(\mathbf{G})$ is a set of elements called vertices (which is coincides with χ) and $E(\mathbf{G})$ is a binary relation on $V(\mathbf{G})$ (i.e., $E(\mathbf{G}) \subseteq V(\mathbf{G}) \times V(\mathbf{G})$) called edges of $\mathbf{G}$. If the direction is imposed on each edge, then we call the graph as a directed graph or a digraph. Here, we assume that the digraph has a loop at every vertex (i.e., $(p, p) \in E(\mathbf{G})$ for each $p \in V(\mathbf{G})$). Moreover, we assume that there is a distance function ρ defined on the set of vertices $V(\mathbf{G})$, and we call it a weighted graph. $\mathbf{G}^{-1}$ is obtained from $\mathbf{G}$ by reversing the direction of edges. Thus, we have

$$E(\mathbf{G}^{-1}) = \{(q,p) : (p,q) \in E(\mathbf{G})\}.$$

The letter $\widetilde{\mathbf{G}}$ denotes the undirected graph obtained from $\mathbf{G}$ by ignoring the direction of edges. It will be more convenient for us to treat $\widetilde{\mathbf{G}}$ as a directed graph for which the set of its edges is symmetric. Under this convention, we have

$$E(\widetilde{\mathbf{G}}) = E(\mathbf{G}) \cup E(\mathbf{G}^{-1}).$$

Further, if p and q are vertices in a graph $\mathbf{G}$, then a path in $\mathbf{G}$ from p to q of length $N \in \mathbb{N}$ is a sequence $\{p_i\}_{i=1}^{N}$ of N vertices such that $p_1 = p$, $p_N = q$ and $(p_i, p_{i+1}) \in E(\mathbf{G})$ for $i = 1, 2, ..., N-1$. A graph $\mathbf{G}$ is connected if a path exists between any two vertices.

Definition 1 [12] Let $(\chi, \rho, \mathbf{G})$ be a metric space endowed with the directed graph $\mathbf{G}$ and S be a self-mapping on χ. Then S preserves the edges of $\mathbf{G}$ if

$$(p,q) \in E(\mathbf{G}) \Longrightarrow (Sp, Sq) \in E(\mathbf{G}).$$

Definition 2 [12] A mapping $S : \chi \to \chi$ is called $\mathbf{G}$-continuous if given $p \in \chi$ and a sequence $\{p_n\}$,
$p_n \to p$ and $(p_n, p_{n+1}) \in E(\mathbf{G})$ for $n \in \mathbb{N}$ imply $Sp_n \to Sp$.

Definition 3 [8] A one-parameter family $\mathcal{F} = \{S(t) : t \geq 0\}$ of mappings from a set χ into itself is said to be a semigroup of mappings if the following conditions are satisfied:

(i) $S(t+s)p = S(t) \circ S(s)p$ for all $t, s \geq 0$ and $p \in \chi$,
(ii) $S(0)p = p$ for all $p \in \chi$. Define the set of all common fixed points of $\mathcal{F}$ as

$$F(\mathcal{F}) = \bigcap_{t \geq 0} F(S(t)).$$

Here, we present the definition of nearly asymptotically nonexpansive mapping for a one-parameter semigroup of mappings.

Definition 4 A family $\mathcal{F} = \{S(t) : t \geq 0\}$ of mappings from a set χ into itself is said to be a one-parameter $\mathbf{G}$-nearly asymptotically nonexpansive semigroup with $\{k(t) : t \geq 0\}$ if the following conditions are satisfied:

(i) $\rho(S(t)p, S(t)q) \leq k(t)[\rho(p,q) + a_t]$ for all $p, q \in \chi$ and $(p,q) \in E(\mathbf{G})$,
(ii) $S(t+s)p = S(t) \circ S(s)p$ for all $t, s \geq 0$ and $p \in \chi$,
(iii) $S(0)p = p$ for all $p \in \chi$,
(iv) for each $p \in \chi$, the mapping

$$t \to S(t)p \quad \text{is} \quad \text{continuous},$$

(v) $t \to k(t) : [0, \infty) \to [0, \infty)$ is continuous and $t \to a_t : [0, \infty) \to [0, \infty)$ is also continuous,

(vi) $k(t) \geq 1$ for all $t \geq 0$ and $\limsup_{t\to\infty} k(t) = 1$, $\lim_{t\to\infty} a_t = 0$.

Note 1 If a_t becomes 0 for all $t > 0$, then it will be a semigroup of **G**-asymptotically nonexpansive mappings. Because of this a_t non-zero for some $t > 0$, this semigroup of mappings is a semigroup of non-Lipschitzian mappings.

Proving the results of the linear domain in the nonlinear domain always remains an intriguing matter. Thanks to their special kind of convex structures, hyperbolic metric spaces provide a natural platform to study the approximation of fixed points. In 1990, Reich and Shafrir [18] introduced hyperbolic metric spaces and studied the iteration process for nonexpansive mappings in these spaces. In 2004, Kohlenbach [15] introduced more general hyperbolic metric spaces. In this paper, we use the definition of hyperbolic metric spaces provided by Kohlenbach [15].

Definition 5 Let (χ, ρ) be a metric space. Then $(\chi, \rho, \mathcal{W})$ is said to be a hyperbolic metric space if the function $\mathcal{W} : \chi \times \chi \times [0, 1] \to \chi$ satisfies the following conditions

(i) $\rho(r, \mathcal{W}(p, q, \alpha)) \leq \alpha\rho(r, p) + (1 - \alpha)\rho(r, q)$,

(ii) $\rho(\mathcal{W}(p, q, \alpha), \mathcal{W}(p, q, \beta)) = |\alpha - \beta|\rho(p, q)$,

(iii) $\mathcal{W}(p, q, \alpha) = \mathcal{W}(p, q, 1 - \alpha)$,

(iv) $\rho(\mathcal{W}(p, q, \alpha), \mathcal{W}(r, w, \alpha)) \leq \alpha\rho(p, r) + (1 - \alpha)\rho(q, w)$, for all $p, q, r, w \in \chi$ and $\alpha, \beta \in [0, 1]$.

The condition (i) in the above definition is the convexity condition. This is very similar to convexity in linear spaces like Banach spaces.

A linear example of hyperbolic metric spaces is a Banach space, and nonlinear examples are Hadamard manifolds, the Hilbert open unit ball equipped with the hyperbolic metric, and the CAT(0) spaces.

The generalization of the definition of uniform convexity in metric spaces was first given by Goebel et al. [10].

Definition 6 Let (χ, ρ) be a hyperbolic metric space. We say that χ is uniformly convex if one has for any $a \in \chi$, and for all $r, \varepsilon > 0$

$$\delta(r, \varepsilon) = \inf\left\{1 - \frac{1}{r}\rho\left(\mathcal{W}\left(p, q, \frac{1}{2}\right), a\right); \rho(p, a) \leq r, \rho(q, a) \leq r, \rho(p, q) \geq r\varepsilon\right\} > 0. \tag{1}$$

If $\{C_n\}$ is decreasing sequence of bounded, nonempty, closed, and convex subsets of χ, then $\bigcap_{n\geq 1} C_n \neq \emptyset$, which is one of the important properties of complete uniformly convex hyperbolic metric spaces given by [14] and known as property (R).

The proof of the following lemma is inspired by the proof of Lemma 4.4 of Bachar et al. [3].

Lemma 1 *Assume that* (χ, ρ) *is a complete uniformly convex hyperbolic metric space endowed with the digraph* $\mathbf{G}$. *Let* C *be a convex nonempty subset of* χ, *which satisfies the property* (R). *Let* $\mathcal{F} = \{S(t)\}_{t \geq 0}$ *be a semigroup of* $\mathbf{G}$*-nearly asymptotically nonexpansive mappings defined on* C. *Let* K *be a nonempty closed convex subset of* C. *Fix* $p_0 \in C$. *Define the function* $\phi : K \to [0, \infty)$ *by*

$$\phi(p) = \limsup_{t \to \infty} \rho(S(t)p_0, p) = \inf_{s \geq 0} \left(\sup_{t \geq s} \rho(S(t)p_0, p) \right).$$

Then, there exists a unique $r \in K$ such that $\phi(r) = \inf_{p \in K} \phi(p)$.

Proof First, let us discuss some of the properties satisfied by ϕ.

(a) For any $s \in [0, \infty)$ and $p, q \in K$, we have

$$\sup_{t \geq s} \rho(S(t)p_0, p) \leq \sup_{t \geq s} \rho(S(t)p_0, q) + \rho(p, q),$$

which implies $|\phi(p) - \phi(q)| \leq \rho(p, q)$. In other words, ϕ is uniformly continuous. In a similar fashion, we have

$$\rho(p, q) \leq \phi(p) + \phi(q)$$

for any $p, q \in K$.

(b) For any $s_1, s_2 \in [0, \infty)$, $p, q \in K$ and $\alpha \in [0, 1]$, set $s = \max\{s_1, s_2\}$, then we have

$$\sup_{t \geq s} \rho(S(t)p_0, \mathcal{W}(p, q, \alpha)) \leq \alpha \sup_{t \geq s_1} \rho(S(t)p_0, p) + (1 - \alpha) \sup_{t \geq s_2} \rho(S(t)p_0, q),$$

which implies

$$\phi(\mathcal{W}(p, q, \alpha)) \leq \alpha \sup_{t \geq s_1} \rho(S(t)p_0, p) + (1 - \alpha) \sup_{t \geq s_2} \rho(S(t)p_0, q).$$

Since s_1 and s_2 are arbitrarily positive numbers, we get

$$\phi(\mathcal{W}(p, q, \alpha)) \leq \alpha\phi(p) + (1 - \alpha)\phi(q),$$

i.e., ϕ is convex.

Set $\phi_0 = \inf_{p \in K} \phi(p)$. For any $n \geq 1$, the set $k_n = \{p \in K; \phi(p) \leq \phi_0 + \frac{1}{n}\}$ is a bounded, closed and convex nonempty subset of K. Since C satisfies the property (R) and K is closed, we conclude that $K_\infty = \bigcap_{n \geq 1} K_n$ is nonempty. Clearly, for any $r \in K_\infty$, we have $\phi(r) = \phi_0$. Therefore, ϕ attains its minimum. Next, we prove that K_∞ is reduced to one point. It follows from part (a) above that if $\phi_0 = 0$, then K_∞ is reduced to one point. Assume otherwise that $\phi_0 > 0$ and $r_1, r_2 \in K_\infty$ with $r_1 \neq r_2$. Let $\eta < \phi_0$. Then, by the definition of ϕ, there exists $s_0 \geq 0$ such that

$$\sup_{t \geq s_0} \rho(S(t)p_0, r_i) < \phi_0 + \eta$$

for $i = 1, 2$. Fix any $t \geq s_0$. Set $R = \phi_0 + \eta < 2\phi_0$. Using the property of uniform convexity, there exists $\delta(r_1, r_2, R) > 0$ such that

$$\rho\Big(S(t)p_0, \mathcal{W}(r_1, r_2, \frac{1}{2})\Big) \leq R(1 - \delta(r_1, r_2, R))$$

for any $t \geq s_0$. Since $R \leq 2\phi_0$, then $\delta(r_1, r_2, R) \geq \delta(r_1, r_2, 2\phi_0) = \delta > 0$, which implies

$$\rho\Big(S(t)p_0, \mathcal{W}(r_1, r_2, \frac{1}{2})\Big) \leq R(1 - \delta) = (\phi_0 + \eta)(1 - \delta)$$

for any $t \geq s_0$. Hence

$$\phi\Big(\mathcal{W}(r_1, r_2, \frac{1}{2})\Big) \leq \sup_{t \geq s_0} \rho\Big(S(t)p_0, \mathcal{W}(r_1, r_2, \frac{1}{2})\Big) \leq (\phi_0 + \eta)(1 - \delta).$$

Since δ is independent from η, we may let $\eta \to 0$ to get

$$\phi_0 \leq \phi\Big(\mathcal{W}(r_1, r_2, \frac{1}{2})\Big) \leq \phi_0(1 - \delta) < \phi_0.$$

This contradiction proves that K_∞ is reduced to one point. Therefore, ϕ has a unique minimum in K as claimed. □

Throughout the paper, we will assume that **G**-intervals are closed and convex. We know that a **G**-interval is any of the subsets

$$[a, \rightarrow) = \{p \in \chi; (a, p) \in E(\mathbf{G})\}$$

and

$$(\leftarrow, b] = \{p \in \chi; (p, b) \in E(\mathbf{G})\}$$

for every $p, q \in \chi$.

3 Main results

The Banach contraction principle states that "Every contraction mapping on a complete metric space has a unique fixed point".

In the following theorem, we generalize the Banach contraction principle. Here, we give a theorem for the existence of a common fixed point of the semigroup of contraction mappings defined on a complete metric space endowed with a digraph.

Theorem 1 *Let* (χ, ρ) *be a complete metric space endowed with the digraph* **G**, *and* $\mathcal{F} = \{S(t)\}_{t \geq 0}$ *be a semigroup of self-mappings on* χ. *Assume that the following conditions hold*:

(a) $(p, S(t)p) \in E(\mathbf{G})$ for all $p \in \chi$,
(b) $S(t)$ preserves the edges of **G**,
(c) $\{S(t)\}_{t \geq 0}$ is **G**-continuous,
(d) there exists $\alpha \in [0, 1)$ such that

$$\rho(S(t)p, S(t)q) \leq \alpha\rho(p, q)$$

for all $p, q \in \chi$ with $(p, q) \in E(\mathbf{G})$. Then, there exists a common fixed point $r \in F(\mathcal{F})$.

Proof. Let $p_0 \in \chi$, then from the condition (a), $(p_0, S(t_0)p_0) \in E(\mathbf{G})$, for any fix $t_0 > 0$. Define the Picard iteration as follows $p_n = S^n(t_0)p_0$ for all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. As $S(t)$ preserves the edges of $\mathbf{G}$, we obtain $(S(t_0)p_0, S^2(t_0)p_0)$, $(S^2(t_0)p_0, S^3(t_0)p_0)$, ..., $(S^n(t_0)p_0, S^{n+1}(t_0)p_0)$, ... $\in E(\mathbf{G})$ so that

$$(p_n, p_{n+1}) \in E(\mathbf{G}), \quad \forall n \in \mathbb{N}_0.$$

Applying the contraction condition, we deduce that

$$\rho(p_{n+1}, p_{n+2}) \leq \alpha^{n+1}\rho(p_0, S(t_0)p_0), \quad n \in \mathbb{N}_0.$$

Using the triangular inequality, we have

$$\begin{aligned}\rho(p_{n+1}, p_{n+p}) &\leq \rho(p_{n+1}, p_{n+2}) + \rho(p_{n+2}, p_{n+3}) + \cdots + \rho(p_{n+p-1}, p_{n+p}) \\ &\leq (\alpha^{n+1} + \alpha^{n+2} + \cdots + \alpha^{n+p-1})\rho(p_0, S(t_0)p_0) \\ &= \alpha^n \rho(p_0, S(t_0)p_0) \sum_{j=1}^{p-1} \alpha^j \to 0 \quad \text{as} \quad n \to \infty,\end{aligned}$$

which implies that the sequence $\{p_n\}$ is Cauchy in χ. As (χ, ρ) is complete, there exists $r \in \chi$ such that

$$p_n \to r.$$

According to the assumption (c), $S(t_0)$ is continuous, we have

$$p_{n+1} = S(t_0)p_n \to S(t_0)r.$$

By the uniqueness of limit, we get $S(t_0)r = r$.

Let us prove that r is a common fixed point of $\mathcal{F}$. We have

$$S^n(t_0) \circ S(t) = S(nt_0 + t) = S(t) \circ S^n(t_0)$$

for any $n \in \mathbb{N}$ and all $t \geq 0$. Hence

$$S(t_0)(S(t)r) = S(t)(S(t_0)r) = S(t)r,$$

that is, $S(t)r$ is also a fixed point of $S(t_0)$. By the assumption (a), we have

$$(r, S(t)r) \in E(\mathbf{G}).$$

Now, we get

$$\rho(r, S(t)r) = \rho(S(t_0)r, S(t_0)S(t)r)$$
$$\leq \alpha\rho(r, S(t)r).$$

This gives $\rho(r, S(t)r) = 0$. Hence $S(t)r = r$ for all $t \geq 0$ as claimed.

Remark 1 The above theorem generalizes the Theorem 3.1 of Bachar et al. [3], where the author took a monotone semigroup of contraction mappings.

If we replace contraction mapping with more general mappings like nonexpansive mapping in the Banach contraction principle, then this principle does not work. In 1965, Browder and Göhde independently proved a fixed point theorem, which states "Every nonexpansive mapping defined on a uniformly convex Banach space has a fixed point".

In the following theorem, we take a hyperbolic metric space as underlying space, which is a nonlinear kind of space, and a semigroup of nearly asymptotically nonexpansive mappings, which is a generalization of nonexpansive mappings.

Theorem 2 *Assume that (χ, ρ) is a complete uniformly convex hyperbolic metric space endowed with the digraph $\mathbf{G}$. Let C be a convex nonempty subset of χ which satisfies the property (R). Let $\mathcal{F} = \{S(t)\}_{t \geq 0}$ be a semigroup of $\mathbf{G}$-nearly asymptotically nonexpansive mappings, where $S(t)$ preserves the edges of $\mathbf{G}$ for each $t \geq 0$. Assume that there exists $p_0 \in C$ such that $(p_0, S(t)p_0) \in E(\mathbf{G})$ (or $(S(t)p_0, p_0) \in E(\mathbf{G})$) for each $t \geq 0$. Then, there exists a common fixed point $r \in F(\mathcal{F})$ such that $(p_0, r) \in E(\mathbf{G})$ (or $(r, p_0) \in E(\mathbf{G})$).*

Proof. Since $(p_0, S(t)p_0) \in E(\mathbf{G})$, then in order to prove

$$\bigcap_{t \geq 0} \Big([S(t)p_0, \rightarrow) \cap C \Big)$$

is a nonempty closed convex subset of C, it is enough to show that

$$\bigcap_{i \in \{1,2,3,\ldots n\}} \Big([S(t_i)p_0, \rightarrow) \cap C \Big) \neq \emptyset$$

for any arbitrary numbers $t_1, t_2, t_3, \ldots, t_n \in [0, \infty)$. Set

$$p = S(t_1 + t_2 + t_3 + \cdots + t_n)p_0 \in C.$$

Since the semigroup is endowed with the binary relation, $(p_0, S(t)p_0) \in E(\mathbf{G})$ and $S(t)$ preserves the edges of $\mathbf{G}$, then we have $(S(s)p_0, S(s+t)p_0) \in E(\mathbf{G})$ for any $s, t \geq 0$. This will imply $(S(t_i)p_0, p) \in E(\mathbf{G})$ for any $i = 1, 2, 3, \ldots, n$, i.e.,

$$p \in [S(t_i)p_0, \rightarrow) \cap C, \quad i = 1, 2, 3, \ldots, n.$$

Hence

$$\bigcap_{i\in\{1,2,3,\ldots,n\}} \Big([S(t_i)p_0, \rightarrow) \cap C\Big) \neq \emptyset$$

holds. Next, we consider the function $\phi : C_\infty \rightarrow [0, \infty)$ defined by

$$\phi(p) = \limsup_{s\rightarrow\infty} \rho(S(2s+t)p_0, p) = \inf_{t+s+\eta\geq 0}\big(\sup_{t+2s\geq\eta+s+t} \rho(S(t+2s)p_0, p)\big),$$

where $s, t, \eta \geq 0$. By Lemma 1, there exists a unique $r \in C_\infty$ such that

$$\phi(r) = \inf\{\phi(p) : p \in C_\infty\}.$$

Now, to prove r is a common fixed point of the semigroup $\mathcal{F}$, let $p \in C_\infty$ and $s, t \geq 0$. If $t \geq s$, then $(S(t-s)p_0, p) \in E(\mathbf{G})$ implies $(S(t)p_0, S(s)p) \in E(\mathbf{G})$. Otherwise, assume $t < s$. Set $\epsilon = s - t$. Since $(p_0, p) \in E(\mathbf{G})$, we get $(p_0, S(\epsilon)p) \in E(\mathbf{G})$, which implies $(S(t)p_0, S(t)S(\epsilon)p) \in E(\mathbf{G})$. Therefore $(S(t)p_0, S(s)p) \in E(\mathbf{G})$ for any $t, s \geq 0$, which implies $S(s)p \in C_\infty$ for any $s \geq 0$. In other words, C_∞ is invariant by the semigroup $\mathcal{F}$. Since $(S(t+s)p_0, r) \in E(\mathbf{G})$, we have

$$\begin{aligned}
\rho(S(s)\circ S(t+s)p_0, S(s)r) &\leq k(s)\rho(S(t+s)p_0, r) + k(s)a_s \\
&\leq \sup_{s\geq\eta}[k(s)\rho(S(t+s)p_0, r) + k(s)a_s] \\
&= \sup_{s\geq\eta} k(s) \sup_{s+t\geq\eta+t} \rho(S(t+s)p_0, r) + \sup_{s\geq\eta} k(s)a_s \\
&= \sup_{s+t\geq\eta+t} \rho(S(t+s)p_0, r).
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\phi(S(s)r) &\leq \sup_{t+2s\geq\eta+s+t} \rho(S(t+2s)p_0, S(s)r) \\
&\leq \sup_{t+s\geq\eta+t} \rho(S(t+s)p_0, r).
\end{aligned}$$

Since η is taken as an arbitrary positive, we obtain $\phi(S(s)r) = \leq \phi(r)$. Lemma 1 forces us that $S(s)r = r$ by the uniqueness of the minimum point in C. Since s is taken as an arbitrary positive, we conclude that $r \in F(\mathcal{F})$ as claimed. Since $(p_0, r) \in E(\mathbf{G})$, then the proof of theorem is completed.

Remark 2 This theorem generalizes the Theorem 4.5 of Bachar et al. [3], where the author took a monotone semigroup of nonexpansive mappings and Theorem 3.1 of Aggarwal et al. [1], in which only one monotone nearly asymptotically nonexpansive mapping is considered.

4 Conclusion

Theorem 1 is an extension of the Banach contraction principle. In this theorem, we have proved the existence of a common fixed point for a semigroup of contraction mappings in a metric space endowed with a digraph.

Theorem 2 is the existence theorem of a common fixed point for a semigroup of **G**-nearly asymptotically nonexpansive mappings in a hyperbolic metric space endowed with a digraph. In this theorem, we have not used the continuity assumption in the hypothesis of the theorem. To the best of our knowledge, we are the first who gave the existence theorem for a semigroup of nearly asymptotically nonexpansive mappings without using continuity assumption.

Let us finish the paper with the following open question:

Open Question: Can we find a non-trivial example of a **G**-nearly asymptotically nonexpansive semigroup in a hyperbolic metric space endowed with a graph?

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Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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