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ON A CLASS OF SOLUTIONS FOR A QUADRATIC DIOPHANTINE EQUATION

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Abstract

Let $P \coloneqq P(x)$ be a polynomial in $Z[x]$. In this manuscript, we think about the integral points of a bilinear Diophantine equation $DE : \alpha^2 - 90\beta^2 - 10\alpha - 1260\beta = 4401$. We as well get hold of a few formulae and recurrence sequence to the integral points of (α_n, β_n) of DE.

1. Introduction

A Diophantine equation is a polynomial equation $P(x_1, x_2, \ldots, x_n) = 0$ where the polynomial P has integral coefficients and one is involved in solutions for which all the unknowns take integer values. For example, $x^2 + y^2 = z^2$ and $x = 3,\ y = 4,\ z = 5$ is one of its infinitely many solutions. A dissimilar example is $x + y = 1$ and every one of its solutions are given by $x = t,\ y = 1 - t$ where t passes all the way through all integers. A third example is $x^2 + 4y = 3$. This Diophantine equation has no solutions,

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although note that $x = 0,\ y = \frac{3}{4}$ is a solution with rational values for the unknowns. Diophantine equations are rich in variety. Two-variable Diophantine equation has been subject matter to general research, and their theory constitutes one of the nearly everyone gorgeous, most elaborate parts of mathematics, which nevertheless still keeps some of its secrets for the next generation of researchers.

In this manuscript, we consider positive integral points of the Diophantine equation $\alpha^2 - 90\beta^2 - 10\alpha - 1260\beta = 4401$ which is malformed into a Pell's equation and is solved by a variety of methods.

2. Preliminaries

Consider the Diophantine equation

$$DE : \alpha^2 - 90\beta^2 - 10\alpha - 1260\beta = 4401 \tag{1}$$

to be solved over Z. It is not trouble-free to work out and find the nature and properties of the solutions of (DE). So we survive relevant a linear conversion Trs to (DE) to reassign to a simpler form for which we can decide the integral points.

Let

$$Trs : \begin{cases} \alpha = x + h \\ \beta = y + k \end{cases} \tag{2}$$

be the conversion where $h, k \in Z$.

Applying Trs to DE, we get

$$T(DE) = \widetilde{D}E : (x + h)^2 - 90(y + k)^2 - 10(x + h) - 1260(y + k) = 4401. \tag{3}$$

Equating the coefficients of x and y to zero, we obtain $h = 5$ and $k = -7$. Therefore for as well as $\alpha = x + 5, \beta = y - 7$, we have the Diophantine equation

$$\widetilde{D}E : x^2 - 90y^2 = 16 \tag{4}$$

which is a Pell equation. Currently, we attempt to discover all integer points

(x_n, y_n) of $\widetilde{DE}$ along with then, we preserve retransfer all consequences from $\widetilde{DE}$ to DE utilizing the converse of Trs.

Theorem 2.1. *Let $\widetilde{DE}$ be the Diophantine equation in* (4). *Then*

(i) *The continued fraction expansion of* $\sqrt{90}$ *is*

$$\sqrt{90} = [9; \overline{2, 18}]$$

(ii) *The primary solution of* $x^2 - 90y^2 = 1$ *is* $(u_1, v_1) = (19, 2)$

(iii) *For* $n \geq 4$,

$$u_n = 39(u_{n-1} - u_{n-2}) + u_{n-3}$$

$$v_n = 39(v_{n-1} - v_{n-2}) + v_{n-3}.$$

Proof.

(i) The continued fraction expansion of $\sqrt{90} = 9 + (\sqrt{90} - 9)$

$$= 9 + \cfrac{1}{\cfrac{1}{\sqrt{90} - 9}}$$

$$= 9 + \cfrac{1}{\cfrac{\sqrt{42} + 6}{6}}$$

$$= 9 + \cfrac{1}{2 + \cfrac{\sqrt{90} - 9}{9}}$$

$$= 9 + \cfrac{1}{2 + \cfrac{1}{\sqrt{90} + 9}}$$

$$= 9 + \cfrac{1}{2 + \cfrac{1}{18 + (\sqrt{90} - 9)}}.$$

Consequently the continued fraction expansion of $\sqrt{90}$ is

$$[9; \overline{2, 18}]$$

(i) Make a note of with the intention of by (3), if $(u_1, v_1) = (19, 2)$ is the primary solution of $x^2 - 90y^2 = 1$, then the supplementary solutions (u_n, v_n) of $x^2 - 90y^2 = 1$ can be derived employing the equalities $(u_n + v_n\sqrt{90}) = (u_1 + \sqrt{90}\,v_1)^n$ for $n \geq 2$, in another words,

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 & 90v_1 \\ 2 & u_1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

For $n \geq 2$. As a outcome it can be given away by generation on$n$ that

$$u_n = 39(u_{n-1} - u_{n-2}) + u_{n-3}$$

and also

$$v_n = 39(v_{n-1} - v_{n-2}) + v_{n-3}$$

for $n \geq 4$.

Now we regard as the predicament

$$x^2 - 90y^2 = 16.$$

Reminder that we represent the integer solutions of $x^2 - 90y^2 = 16$ by (x_n, y_n), and stand for the integer points of $x^2 - 90y^2 = 1$ by (u_n, v_n) afterward we encompass the subsequent theorem.

Theorem 2.2. *Identify a progression* $\{(x_n, y_n)\}$ *of positive integers by*

$$(x_1, y_1) = (76, 8)$$

and

$$x_n = 76u_{n-1} + 720v_{n-1}$$

$$y_n = 8u_{n-1} + 76v_{n-1},$$

where $\{(u_n, v_n)\}$ *is a string of positive solutions of* $x^2 - 90y^2 = 1$. *Then*

(1) (x_n, y_n) *is a solution of* $x^2 - 90y^2 = 16$ *for every integer* $n \geq 1$.

(2) *For* $n \geq 2$

$$x_{n+1} = 19x_n + 180y_n$$

$$y_{n+1} = 2x_n + 19y_n.$$

(3) *For* $n \geq 4$

$$x_n = 39(x_{n-1} - x_{n-2}) + x_{n-3}$$

$$y_n = 39(y_{n-1} - y_{n-2}) + y_{n-3}.$$

Proof.

(1) It is straightforwardly seen that

$$(x_1, y_1) = (76, 8)$$

Is a explanation of $x^2 - 90y^2 = 16$ agreed that

$$x_1^2 - 90y_1^2 = (76)^2 - 90(8)^2$$

$$= 16(19^2 - 90(2^2))$$

$$= 16(1)$$

$$= 16.$$

Make a note of that by description (u_{n-1}, v_{n-1}) is an explanation of $x^2 - 90y^2 = 1,$ that is,

$$u_{n-1}^2 - 90v_{n-1}^2 = 1. \tag{7}$$

Moreover, we see as greater than that (x_1, y_1) is a solution $x^2 - 90y^2 = 16,$ of that is,

$$x_1^2 - 90y_1^2 = 16. \tag{8}$$

Applying, (7) and (8), we get a hold

$$x_n^2 - 90y_n^2 = (76u_{n-1} + 720v_{n-1})^2 - 90(8u_{n-1} + 76v_{n-1})^2$$

$$= u_{n-1}^2(2^4) - v_{n-1}^2(2^4(90))$$

$$= 2^4(u_{n-1}^2 - 90v_{n-1}^2)$$

$$= 2^4.$$

Therefore (x_n, y_n) is a solution of $x^2 - 90y^2 = 2^4$.

(2) Remember that $x_{n+1} + y_{n+1}\sqrt{d} = (u_1 + v_1\sqrt{d})(x_n + y_n\sqrt{d})$

Therefore $x_{n+1} = u_1x_n + v_1y_nd$ and $y_{n+1} = v_1x_n + u_1y_n$

So $x_{n+1} = 19x_n + 180y_n$ and $y_{n+1} = 2x_n + 19y_n$

Because $u_1 = 19$ and $v_1 = 2$.

(3) Applying the equalities

$$x_n = 2^2(19)u_{n-1} + 2^3(90)v_{n-1} \text{ and } x_{n+1} = 19x_n + 180y_n.$$

We come across by generation on n that

$$x_n = 39(x_{n-1} - x_{n-2}) + x_{n-3}$$

for $n \geq 4$.

Similarly, it can be shown that

$$y_n = 39(y_{n-1} - y_{n-2}) + y_{n-3}.$$

We saw as above that the Diophantine equation DE could be distorted into the Diophantine equation $\widetilde{DE}$ via the makeover Trs. As well, we showed that $\alpha = x + 5$ and $\beta = y - 7$. So we can retransfer all outcome from $\widetilde{DE}$ to by using the inverse of Trs. Consequently we can give the subsequent foremost theorem.

Theorem 3.1. *Let DE be the Diophantine equation in* (1). *Then*

(1) *The primary solution of DE is* $(\alpha_1, \beta_1) = (81, 1)$.

(2) *Characterize the progression* $\{(\alpha_n, \beta_n)\}_{n\geq 1} = \{(x_n + 5, y_n - 7)\}$, *where* $\{(x_n, y_n)\}$ *given in Theorem* 2.2 *at that time* (α_n, β_n) *is a clarification of DE. As a result it has infinitely many points* $(\alpha_n, \beta_n) \in Z \times Z$.

(3) *The points* (α_n, β_n) *convince*

$$\alpha_n = 19\alpha_{n-1} + 180\beta_{n-1} + 1170$$

$$\beta_n = 2\alpha_{n-1} + 19\beta_{n-1} + 116.$$

(4) *The points* (α_n, β_n) *gratify the recurrence relationships*

$$\alpha_n = 39(\alpha_{n-1} - \alpha_{n-2}) + \alpha_{n-3} \ \beta_n = 39(\beta_{n-1} - \beta_{n-2}) + \beta_{n-3}.$$

4. Conclusion

Diophantine equationsare well-to-do in assortment. Present is refusal worldwide technique for discovery all achievable solutions (if it exists) for Diophantine equation. The technique looks to be straightforward but it is extremely easier said than done for attainment the solutions.

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