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The impracticality of homogeneously weighted moving average and progressive mean control chart approaches

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Abstract
There is growing literature on new versions of "memory-type" control charts, where deceptively good zero-state average run-length (ARL) performance is misleading. Using steady-state run-length analysis in combination with the conditional expected delay (CED) metric, we show that the increasingly discussed progressive mean (PM) and homogeneously weighted moving average (HWMA) control charts should not be used in practice. Previously reported performance of methods based on these two approaches is misleading, as we found that performance is good only when a process change occurs at the very start of monitoring. Traditional alternatives, such as exponentially weighted moving average (EWMA) and cumulative sum (CUSUM) charts, not only have more consistent detection behavior over a range of different change points, they can also lead to better out-of-control zero-state ARL performance when properly designed.

KEYWORDS
average run length, control chart, statistical process monitoring, steady-state average run length

1 | INTRODUCTION

The statistical process monitoring (SPM) field comprises many different procedures to detect quickly and reliably changes in a process based on sequentially observed data. There are classical ones such as the Shewhart chart,[1] Shewhart charts with runs rules,[2] the cumulative sum (CUSUM) chart,[3] the exponentially weighted moving average (EWMA) chart,[4] and the less popular Shiryaev-Roberts scheme.[5] These methods are described in textbooks on statistical quality control like Montgomery,[6] Ryan,[7] Kenett et al,[8] and Vardeman and Jobe,[9] to name a few. In addition to some more or less straightforward modifications, such as the combination charts of Lucas,[10, 11] schemes with adaptive parameters like Capizzi and Masarotto,[12] and generalized likelihood ratio approaches (unspecified out-of-control mean), for example, Capizzi,[13] we have been facing for the last 20 years some unfortunate developments. Quite a substantial set of charts have been proposed, which share one common feature: they often exhibit excellent out-of-control zero-state average run-length (ARL) performance, which implies quick detection of changes that happen right at the beginning of the monitoring, but their detection performance deteriorates substantially when the process change happens later. The balance between detection of very early change points and later ones is ignored, an issue that has been thoroughly discussed in many papers, including Chandrasekaran et al,[14] Knoth,[15] Lucas and Saccucci,[11] and Woodall.[16]

The most prominent example of this unfortunate phenomenon is the synthetic chart introduced in Wu and Spedding.[17] It has become one of the most popular control chart construction principles during the last 20 years. In Davis and Woodall[18] and later in Knoth,[19] it was made clear, however, that the supposed benefits of synthetic charts are highly questionable due to the presence of an implicit fast initial response (or head-start) feature and a focus on zero-state performance. Despite this, synthetic charts stayed popular and led to many more publications, refer to Rakitzis et al[20] for a vigorous review encouraging further contributions on the topic. Here, we pick two other design strategies that should be avoided in SPM. We discuss the so-called progressive mean (PM) chart introduced by Abbas et al[21] and the closely related homogeneously weighted moving average (HWMA) control chart proposed by Abbas.[22] We also advise against using double and triple EWMA charts (DEWMA and TEWMA, respectively).

The PM chart corresponds to a well-known procedure in sequential statistics,[23–25] utilized for tests of power one and the repeated significance test. In parallel to Abbas et al,[21] Morais et al[26,27] investigated this control chart in some detail. They expressed it as the limit (therefore, they called it the limit chart) of an EWMA chart, where the smoothing constant approaches zero. Moreover, they observed that the major flaw of the PM chart, as we illustrate in Section 2, is that it experiences poor performance in detecting process changes that are delayed from the start of monitoring. Even the originators of the PM charts, as in Abbas et al (p. 625),[28] noticed this situation, writing, *"However, if the process shift occurs at some other time, the PM chart will not be so good."* In Abbas et al,[28] the PM principle was transferred to variance monitoring and labeled as a floating control chart.

Another offspring of the PM chart is the HWMA chart.[22] The HWMA chart combines, using different weights, the current observation with the previous PM. This can attenuate the inertia resulting from the use of the past data,[29] but it does not eliminate the problem, as seen in Section 3 with a nonmonotonic behavior of the ARL at different change points.

Despite their severe disadvantage in performing poorly to detect changes that happen later than at start-up, the PM and HWMA approaches had been used extensively to develop new methods in the SPM literature. Therefore, we want to illustrate clearly that both PM and HWMA approaches should not be applied for process monitoring. To overcome the misleading use of the zero-state ARL, we consider the conditional expected delay (CED),[30] whose limit is the conditional steady-state ARL as the time of the process shift increases.[31–33]

In the next two sections, we evaluate the PM and the HWMA chart, respectively, regarding their CED behavior. In Section 4, we describe the data weighting profiles resulting from these different approaches to illustrate the cause of their poor and misleading behavior. We end by collecting our conclusions in Section 5.

2 | PM APPROACH

2.1 | Basic definitions

Similar to Abbas et al,[21] we assume that $X_1, X_2, \ldots$ follow a normal distribution with mean μ and standard deviation σ. Moreover, the values of X_t are assumed to be stochastically independent. Because we are interested in changes that might happen after $t = 1$, we introduce the following change point (τ) model

$$\mu = \begin{cases} \mu_0 = 0, \ t < \tau \\ \mu_1 = \delta, \ t \geq \tau. \end{cases} \tag{1}$$

Later we will use the change point τ as an index to label a particular change point position, as needed. The standard deviation is assumed to be known, $\sigma = \sigma_0 = 1$ (otherwise normalize the X_t), and to remain constant. Then the PM chart is formed by

$$P_t = \frac{1}{t} \sum_{i=1}^{t} X_i \quad , \ t = 1, 2, \ldots$$

with

$$E_\infty(P_t) = \mu_0 \quad \text{and} \quad \text{var}(P_t) = 1/t$$

and run length

$$\tilde{L}_P = \min \left\{ t \geq 1 : |P_t - \mu_0| > 3 c_P \sigma_0 / \sqrt{t} \right\}.$$

TABLE 1 Excerpt of zero-state ARLs from Table II in Abbas et al[21] (10^4 replications), $\text{ARL}_0 = 500$, new Monte Carlo results, and two EWMA configurations evaluated numerically

	$\delta =$ 0	0.25	0.5	0.75	1	1.5	2	2.5	3	4	5
	ARL_δ for PM with Monte Carlo, number of replications =										
10^4 in Abbas et al[21]	498.14	47.24	19.03	11.16	7.55	4.52	3.15	–	1.98	1.42	1.12
10^8	497.89	47.68	19.00	11.07	7.58	4.50	3.16	2.43	1.98	1.43	1.12
	ARL_δ for EWMA										
$\lambda = 0.1$	500.00	103.32	28.81	13.61	8.21	4.17	2.66	1.92	1.51	1.12	1.01
$\lambda = 0.007$	500.00	45.74	14.91	7.66	4.83	2.62	1.78	1.38	1.18	1.02	1.00

Abbas et al21 observed that the above limits are quite wide for large t. They proposed to bend the limits by manipulating the power of t, namely,

$$L_P = \min\left\{t \geq 1 : |P_t - \mu_0| > 3c_P\sigma_0/t^{0.7}\right\} \tag{2}$$

replacing the original $t^{0.5}$ with $t^{0.7}$.

Simple algebra yields the standard recursive formula for the sample mean

$$P_t = \frac{1}{t}X_t + \frac{t-1}{t}P_{t-1}\,,$$

which was explicitly discussed in Abbas,34 who claimed that the PM chart is a special case of the adaptive EWMA (AEWMA) chart of Capizzi and Masarotto.[12] However, this is misleading because the latter method adapts the smoothing constant differently while aiming at a different goal. If the distance between the current observation and the previous AEWMA statistic becomes large, the scheme switches to "Shewhart" mode in a smooth way. Hence, the weights are controlled by the data. The weights of the PM chart are following a deterministic data-free scheme instead. As well, the method of Han and Tsung[35] is only weakly related to the PM chart. These authors pick an optimal EWMA weight from the grid $\{1, 1/2, \dots, 1/t\}$ for each time point t making the magnitude of the resulting EWMA statistic maximal. Here, the PM weight $1/t$ would be the extreme choice. In sum, Capizzi and Masarotto[12] and Han and Tsung[35] proposed EWMA modifications providing a good detection performance for a range of possible changes δ. In contrast, the PM chart exhibits a weighting scheme that is suitable for detecting very early changes, whereas it deteriorates quickly and substantially for later changes.

2.2 | ARL types

In (2), the constant c_P is set so that the in-control zero-state ($\tau = \infty$) ARL, $E_\infty(L_P)$, is equal to some given large value ARL_0. Utilizing Monte Carlo simulation (10^4 replications), Abbas et al[21] determined $c_P = 1.267$ to fulfill $\text{ARL}_0 = 500$. Afterward, they calculated, in the same way, several out-of-control ARL values $E_1(L_P)$ with $\delta \in \{0.25, 0.5, \dots, 5\}$ and concluded that the PM charts perform well, compare to Table 1. Because the simple case $\tau = 1$ in (1) is only one facet of the detection performance, we consider as well the CED

$$D_\tau = E_\tau(L - \tau + 1 \mid L \geq \tau)$$

and, if appropriate, the conditional steady-state ARL

$$\mathcal{D} = \lim_{\tau\to\infty} D_\tau\,.$$

Note that both the sequence of CED values $\{D_\tau\}$ and the limit $\mathcal{D}$ are functions of the shift size δ. For most of the conventional control charts, the sequence converges rapidly to $\mathcal{D}$. Therefore, the conditional steady-state ARL $\mathcal{D}$ is another valuable and representative performance measure. Later we will show that for PM and HWMA charts, more elaborate convergence patterns of $\{D_\tau\}$ appear, so that $\mathcal{D}$ might be less representative. We will estimate all these values with Monte Carlo simulation runs (10^6 replications for each τ). Readers can refer to Siegmund,[36] who did some more sophisticated analyses with the PM chart (under the name repeated significance test or test of power one).

2.3 | EWMA, the classical competitor

Here, we introduce our competitor, the popular EWMA chart,[4] with exact control limits. As MacGregor and Harris[37] mentioned, these limits convey some fast initial response features. Thus, we apply

$$Z_0 = \mu_0,\ Z_t = (1-\lambda)Z_{t-1} + \lambda X_t,\ t = 1, 2, \dots\ ,$$

$$L_E = \min\left\{ t \geq 1 : |Z_t - \mu_0| > c_E \sigma_0 \sqrt{(1-(1-\lambda)^{2t})\frac{\lambda}{2-\lambda}} \right\}.$$

Contrary to PM and HMWA approaches, numerical routines are established to calculate the zero-state ARL, the CED, and the steady-state ARL. Here, we make use of the implementations in the R package `spc`.[38]

2.4 | ARL comparison

Let us start with Table 1, with some confirmation of the zero-state ARL values of Table II in Abbas et al.[21] We use $\text{ARL}_0 = 500$ (and $c_P = 1.267$) and add new Monte Carlo results with 10^8 replicates.

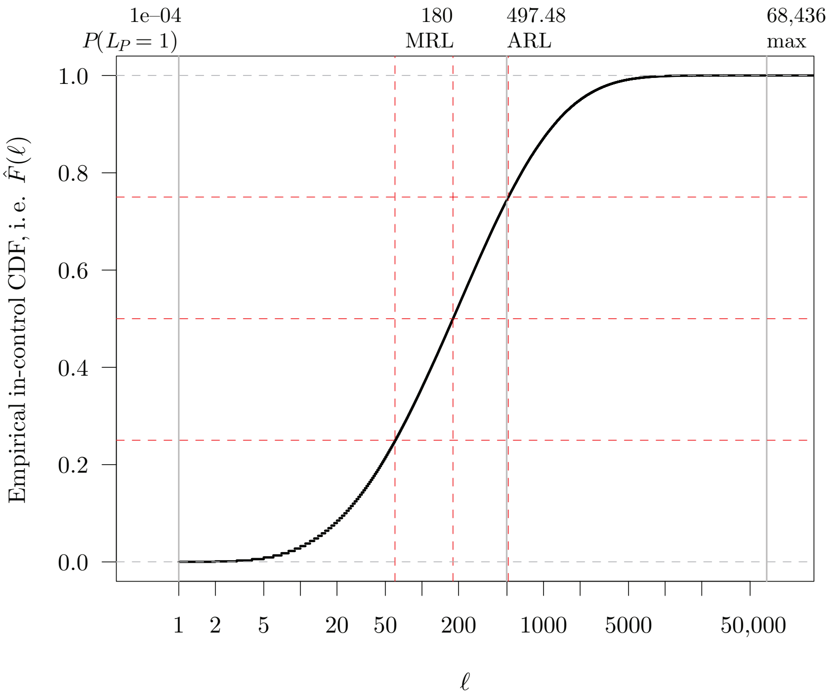
Comparing our results and the previous ones, we verify their accuracy. Abbas et al[21] determined the standard deviation and some percentiles of the run length as well. However, all of these results were obtained for the special situation that the change happens at $\tau = 1$ or never ($\tau = \infty$, i. e., in-control). These ARL results were compared with numbers taken from Lucas and Saccucci[11] for EWMA $\lambda \in \{0.1, 0.25, 0.5, 0.75\}$ and CUSUM ($k = 0.5$), both with and without head-start. Because the PM chart corresponds to an EWMA chart with $\lambda \to 0$, a smaller λ than 0.1 might be more appropriate for the performance comparison. Therefore, we add zero-state ARL results for the EWMA chart with $\lambda \in \{0.1, 0.007\}$. Not surprisingly, one can select an EWMA chart design that beats the PM chart uniformly in terms of the zero-state ARL (see Section 4), but we do not recommend extremely small smoothing constants due to resulting inertia issues.

Before turning to our CED analysis, we consider the empirical cumulative distribution function (CDF) of the in-control run length. The empirical CDF in Figure 1 features two particular patterns for PM chart different from most other control charts. The median run length, MRL = 180, is much smaller than the ARL = 497. In fact, the ARL coincides with the third quartile. The maximum observed run length in our simulation, 68,436, was unusually large.

In order to assess the CED, we calculated, using Monte Carlo (10^6 replicates) and the numerical methods in the R package `spc`, the series $\{D_\tau\}$ for $\tau = 1, 2, \dots, 500 = \text{ARL}_0$ for the PM and EWMA charts. We consider $\delta \in \{0.25, 0.5, 0.75, 1, 2, 2.5, 3, 5\}$ and provide the graphs in Figures 2 and 3. Because D_τ does not converge within $\tau \in \{1, 2, \dots, 500\}$ (in contrast to both EWMA designs), we added on the right-hand side of all diagrams the value D_{1000} in parentheses to get an idea about the progression. We compared the PM chart with two different EWMA configurations, $\lambda = 0.1$ (a standard setup) and $\lambda = 0.007$ (less common, but beats the PM chart uniformly for $\delta \geq 0.25$). Only for the smallest shift considered, $\delta = 0.25$, does the PM chart exhibit a lower D_τ profile as long as $\tau \leq 500$ (which might be sufficient for an in-control ARL of 500). However, for later changes ($\tau > 500$), the common $\lambda = 0.1$ EWMA chart detects the change more quickly on average. For all other δ values, this standard design performs far better than the PM chart. For $\delta > 0.25$, the special $\lambda = 0.007$ EWMA chart has lower and thus better D_τ profiles than the PM chart, in addition to EWMA's zero-state ARL superiority for all δ. Beginning with $\delta = 0.5$, its values fall between those of the more standard EWMA chart and the PM chart.

In conclusion, the out-of-control performance of the PM chart is easily dominated by control charts from the EWMA family.

FIGURE 1 PM—in-control CDF $F(\ell) = P(L_P \leq \ell)$, Monte Carlo with 10^6 replicates



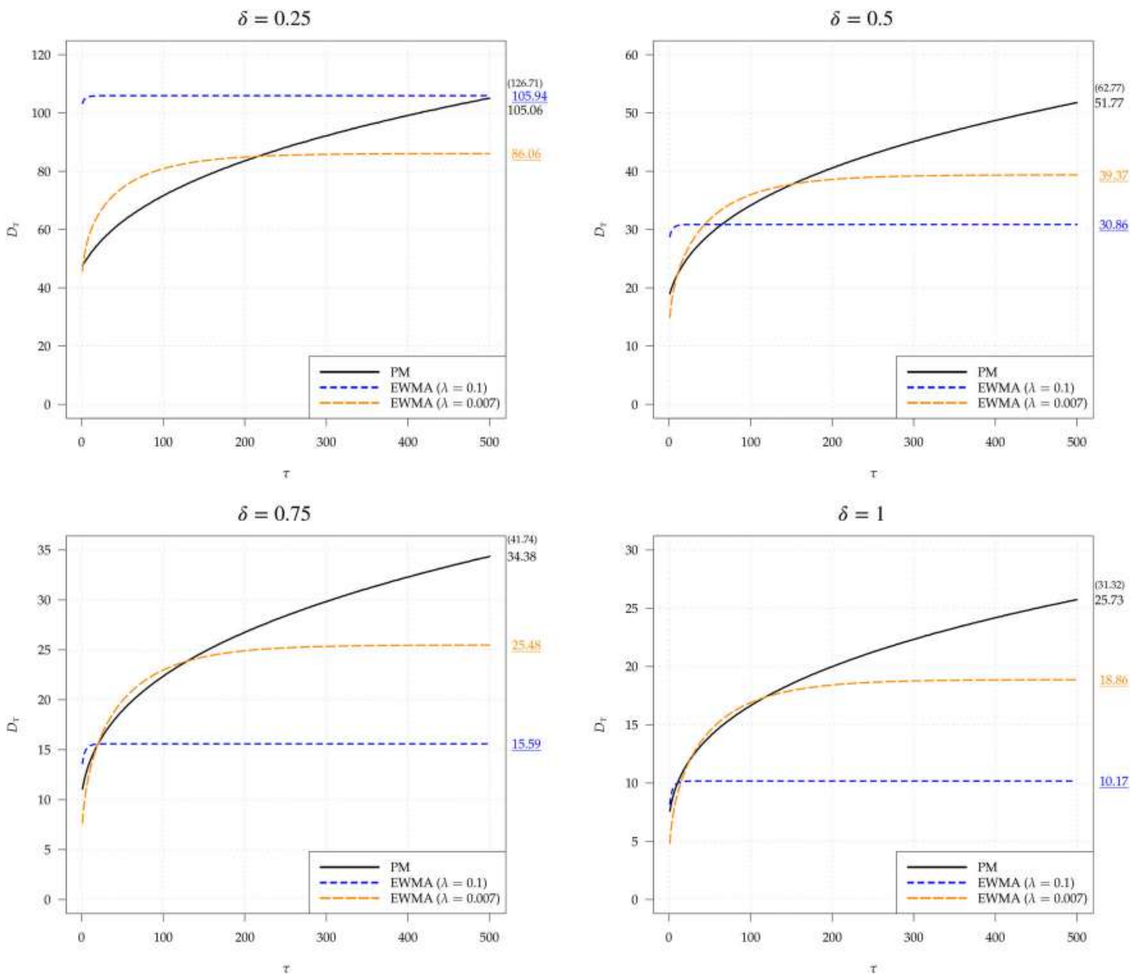


FIGURE 2 CED D_τ ($\delta \in \{0.25, 0.5, 0.75, 1\}$), Monte Carlo with 10^6 replicates, PM chart, two EWMA charts

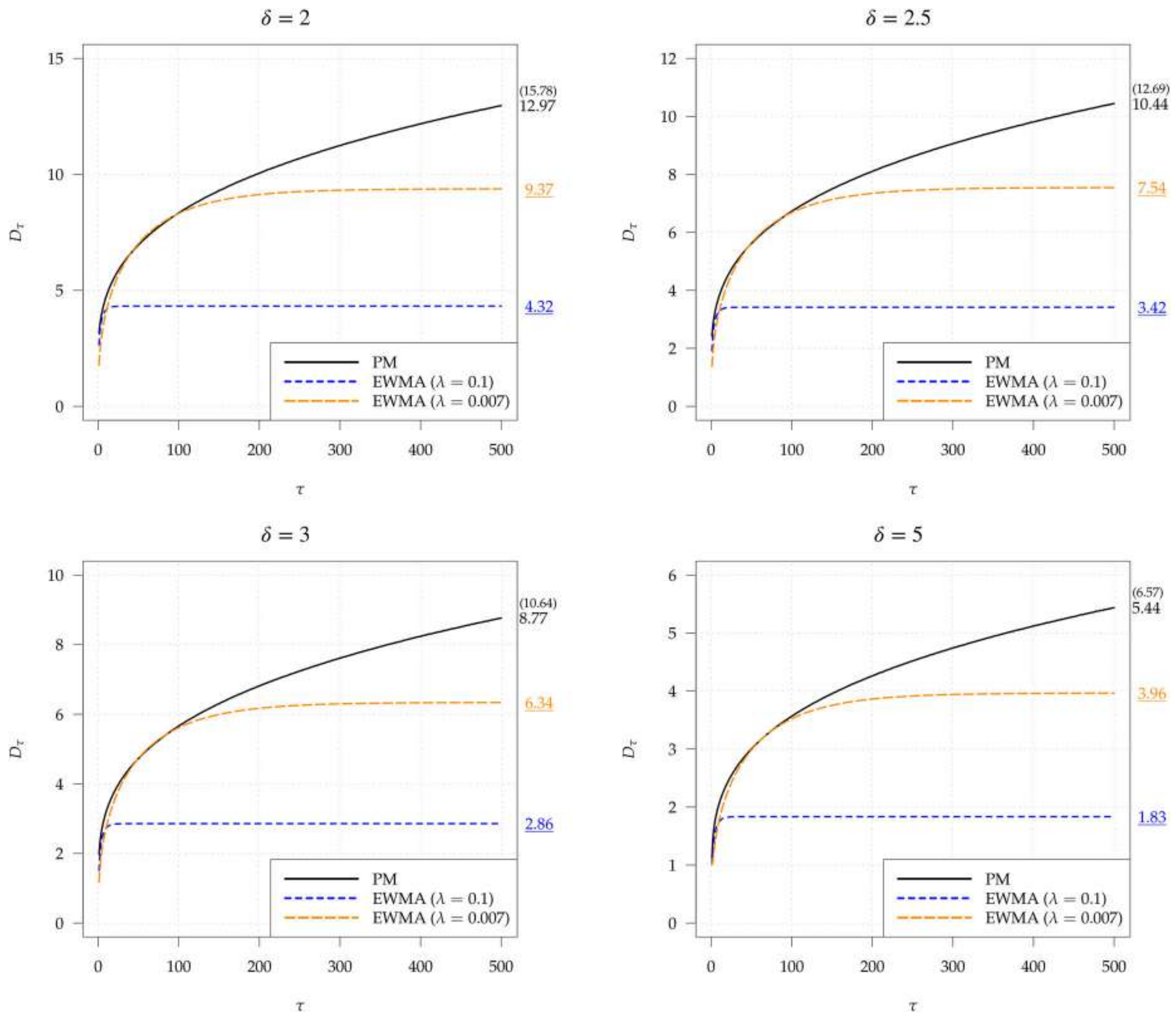


FIGURE 3 CED D_τ ($\delta \in \{2, 2.5, 3, 5\}$), Monte Carlo with 10^6 replicates, PM chart, two EWMA charts

2.5 | Other PM research

In addition to Abbas et al[21] and Abbas,[34] there has been a massive growth of PM modeling and its application in the SPM literature. All of the methods we list below possess the same basic flaw, and thus their practical performance will be poor. We review this literature to show how much work on the PM approach has appeared in the SPM literature.

Nonparametric approaches can be found in Abbasi et al,[39] where they proposed a nonparametric chart based on the PM statistic to monitor the proportion of observations over a target value. Abbas et al[40] suggested the monitoring of a sign statistic using a PM scheme, while Abbas et al[41] repeated the analysis with double PM (DPM) approach (see below). Ali et al[42] combined the EWMA scheme with a PM approach using the sign statistic.

To deal with attribute data, Abbasi[43] used the PM approach to monitor with a series of Poisson observations. Alevizakos and Koukouvinos[44] applied the approach to the Conway–Maxwell–Poisson distribution. Alevizakos and Koukouvinos[45] did something similar, but with the traditional Poisson model. In addition, Abbas et al[46] monitored the fraction of nonconforming items with a PM statistic.

For normal observations, Abbas et al[47] extended the analysis with the DPM chart, based on a PM of progressive means, which adds even more weight to the older observations (see Section 4). However, Riaz et al[48] found an error and corrected the DPM variance to improve its performance. Focusing on a different problem, Zafar et al[49] proposed a chart to monitor the variance. It follows the same spirit as the PM, but the cumulative variance is used instead.

Abbas et al[50] proposed the mixed structure of EWMA and PM charts and named it the mixed EWMA-progressive control chart for detecting small and moderate deviations in process location. Again, Alevizakos et al[51] gave a note on their variance, which was later used by Ajadi et al[52] as a corrected EWMA-PM method. Thereafter, Abbas et al[50] corrected their approach for variance estimation for the same EWMA-PM control chart. Zafar et al [53] developed the MAX-PM control chart similar to the MAX-EWMA approach. The MAX-operation combines two (or more) single control charts (here PM) in order to monitor more than one parameter.

Dealing with linear profiles, Abbas et al[54] used the PM approach to monitor a variable in the presence of an auxiliary variable. In addition, Saeed et al[55] proposed a PM approach for the simultaneous monitoring of simple linear profile parameters named PM_3, under shifts in linear profile parameters such as intercept, slope, and error variance.

With other continuous distributions for monitoring time between events, Alevizakos and Koukouvinos [56] adapted the PM statistic for the Erlang distribution, and Alevizakos and Koukouvinos[57] repeated the analysis but using the DPM approach. Finally, Zafar and Riaz[58] recognized the inertia[29] problem of memory type charts and proposed to restart the plotting statistic at fixed times. They explored this idea with a modification of the CUSUM, EWMA, and PM statistics. Not surprisingly, an improvement was found with the PM chart. However, to force restart a chart every 10th observation made CUSUM and EWMA chart performance worse for changes of 1.75 standard deviations or less, and improving only slightly the average run length for larger changes. This likely happened as a consequence of the already resetting behavior of the CUSUM and the forgetting factor of the EWMA chart, existing elements that already mitigate inertia. In addition, Zafar and Riaz[58] own experimentation showed a CUSUM and an EWMA chart with consistently smaller ARLs in every scenario, which makes the PM statistic unnecessary.

A number of these papers contained errors that were later corrected. For instance, Abbas et al[47] upgraded the existing PM charts by introducing the DPM control charts for normal or unspecified distributions based on computing PM averages twice. However, they introduced the wrong variance formula (after their eq. (15)[47]). In a more recent paper, Riaz et al[48] provided corrections to the variance formula of DPM. Alevizakos and Koukouvinos[45] extended the idea of the PM statistic introducing a second PM statistic for use with the Poisson distribution in Abbasi et al.[43] They suggested the DPM and the optimal DPM charts for enhancing the detection ability of the PM chart. They used the variance formula from Abbas et al.[47] Latter, Abbas et al[59] corrected the variance derivation of this model.

It should be noted that several papers in this area addressed steady-state performance[40, 43, 44 46]; however, their analysis was limited to performance with a change point after 150 observations,[56] in the best cases, and up to 15 in the worst cases.[46] The extent of these analyses is acceptable when discussing an EWMA chart, but they can be misleading for the PM family.

3 | THE HWMA CHART

3.1 | Basic definition

One spin-off of the PM chart is the HWMA control chart proposed by Abbas.[22] Its statistics are calculated in the following way:

$$H_1 = \omega X_1 + (1-\omega)\mu_0 \,.$$

$$H_t = \omega X_t + (1-\omega)P_{t-1} = \omega X_t + (1-\omega)\frac{1}{t-1}\sum_{i=1}^{t-1} X_i = \omega X_t + \sum_{i=1}^{t-1}\frac{1-\omega}{t-1}X_i$$

for $t = 2, 3, \ldots$, where $0 < \omega < 1$. Determining the first two moments, the signaling rule is derived as follows:

$$E_\infty(H_t) = \mu_0,\ \mathrm{var}(H_1) = \omega^2,\ \mathrm{var}(H_t) = \omega^2 + (1-\omega)^2/(t-1),\ t = 2, 3, \ldots \,.$$

$$\sigma_{H,t} = \sqrt{\mathrm{var}(H_t)}\,,$$

$$L_H = \min\left\{t \ge 1 : |H_t - \mu_0| > c_H \sigma_{H,t}\right\}.$$

TABLE 2 Excerpt of zero-state ARL values from tab. 2 in Abbas[22] (10^5 replications), $\text{ARL}_0 = 500$, new Monte Carlo results, and two EWMA configurations evaluated numerically

	$\delta =$ 0	0.25	0.5	0.75	1	1.5	2	2.5	3	4	5
	ARL_δ for HWMA ($\omega = 0.1$) with Monte Carlo, number of replications =										
10^5 in Abbas[22]	499.78	81.48	28.61	14.85	9.35	4.98	3.32	2.45	1.87	–	1.03
10^8	500.45	81.59	28.56	14.88	9.33	4.97	3.32	2.45	1.87	1.21	1.02
	ARL_δ for EWMA										
$\lambda = 0.1$	500.00	103.32	28.81	13.61	8.21	4.17	2.66	1.92	1.51	1.12	1.01
$\lambda = 0.05$	500.00	77.76	23.71	11.87	7.31	3.77	2.43	1.77	1.41	1.09	1.01

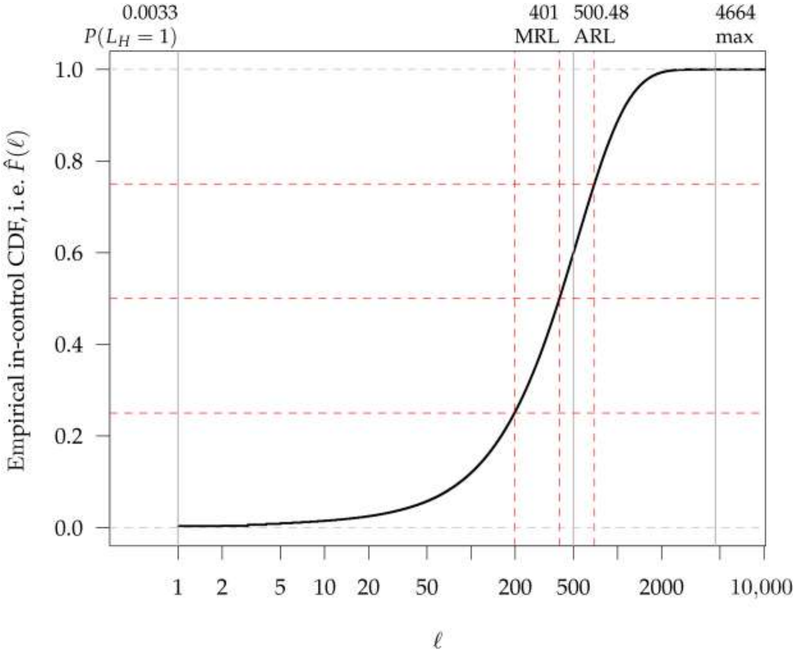


FIGURE 4 HWMA ($\omega = 0.1$)—in-control CDF $F(\ell) = P(L_H \le \ell)$, Monte Carlo with 10^6 replicates

3.2 | ARL comparison

Again, as seen in Table 2, we start with the confirmation of some zero-state ARL results. Here, we pick the case $\omega = 0.1$ and nominal $\text{ARL}_0 = 500$ ($c_H = 2.938$) and collect some out-of-control zero-state ARL results from Table 2 in Abbas,[22] who utilized 10^5 Monte Carlo replications. We added results for two EWMA charts. It turns out that for all EWMA configurations with $\lambda \le 0.05$, the out-of-control zero-state ARL values are uniformly smaller than for the HWMA chart with $\omega = 0.1$. It remains unclear why Abbas[2] compared the HWMA chart with $\omega = 0.1$ to the EWMA chart with $\lambda = 0.1$ since the meanings of the two constants ω and λ are substantially different. A rough calibration could be done by setting equal the asymptotic variances of the statistics H_t and Z_t, that is, $\omega^2 = \lambda/(2-\lambda)$ resulting in $\lambda = 2\omega^2/(\omega^2+1)$. For example, $\omega = 0.1$ would then correspond to $\lambda \approx 0.02$. From Table 2, we conclude that even the use of $\lambda = 0.05$ would lead to a convincing competitor to the HWMA chart with $\omega = 0.1$. From this restricted zero-state ARL analysis one would conclude that the HWMA chart performance is worse than that of the EWMA chart if the EWMA configuration is chosen with some care.

Next, we provide the same illustrations of steady-state performance as in Section 2 for the PM chart. We begin with Figure 4 showing the in-control CDF for the HWMA chart with $\omega = 0.1$. It turns out that the in-control CDF looks similar to CDF profiles for common control charts like the EWMA chart. For the subsequent CED analysis, looking at Figures 5 and 6, we chose two EWMA chart designs, $\lambda = 0.1$ and $\lambda = 0.05$. In the case of small changes, $\delta \le 1$, the EWMA ($\lambda = 0.05$) chart's values D_τ are smaller for each τ considered in Figure 5. However, except for $\delta = 0.25$ and $\tau \le 70$, the more common EWMA chart with ($\lambda = 0.1$) detects the change more quickly on average than the HWMA ($\omega = 0.1$) chart. Turning to the larger changes, in Figure 6, we observe slightly different behavior. For $\delta = 2$ and $\delta = 2.5$, the EWMA ($\lambda = 0.1$) chart is the

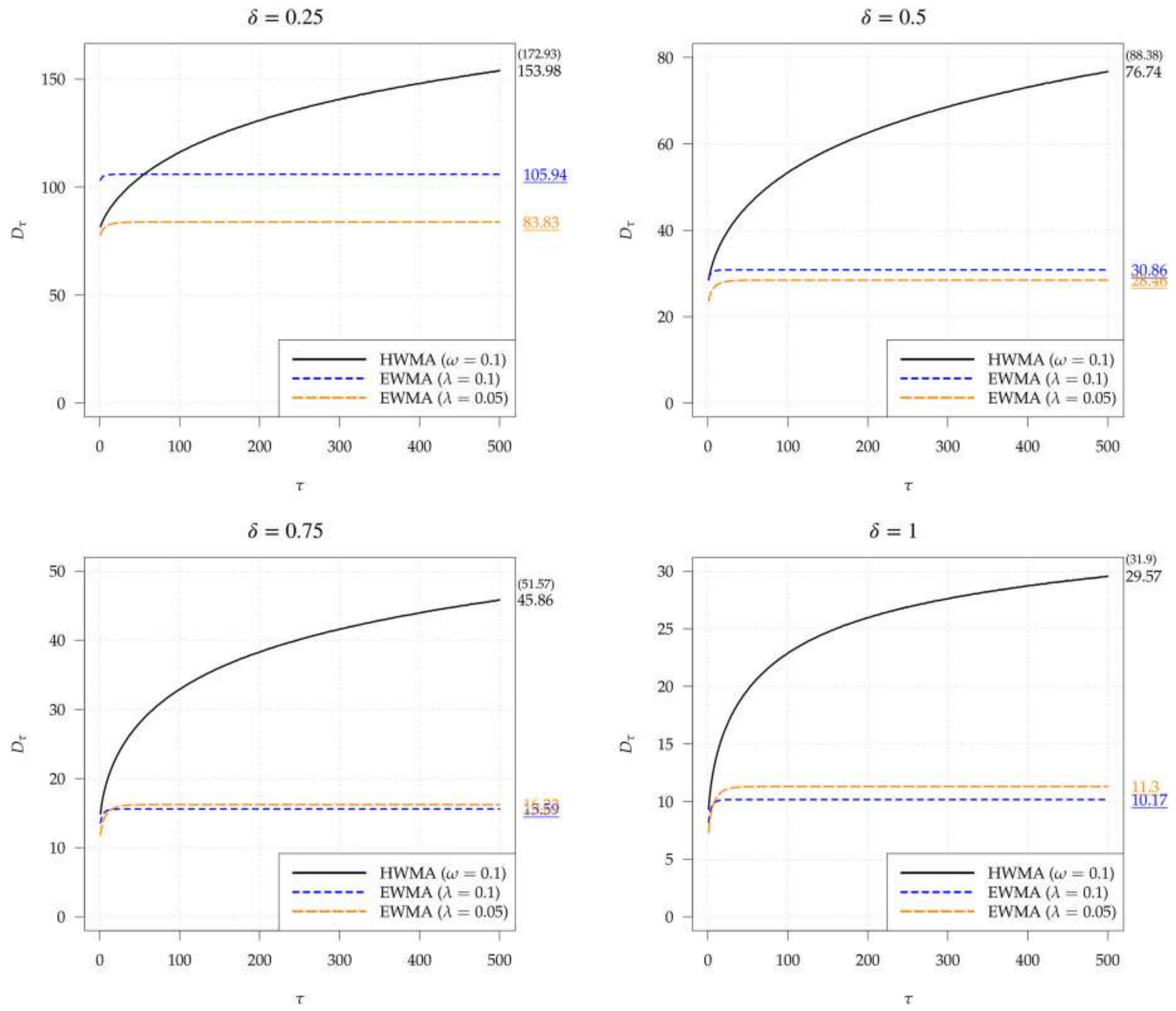


FIGURE 5 CED D_τ ($\delta \in \{0.25, 0.5, 0.75, 1\}$), Monte Carlo with 10^6 replicates, HWMA chart, two EWMA charts

best, followed by the other EWMA design. For $\delta = 2.5$, the HWMA D_τ chart profile starts decreasing for $\tau > 53$, crossing the EWMA ($\lambda = 0.05$) profile around $\tau = 300$. The overall performance, however, is better for the EWMA chart. In case of $\delta = 3$ and even greater values of $\delta(= 5)$, we recognize that the EWMA charts are better for $\tau \leq 120$ ($\lambda = 0.05$) and $\tau \leq 220$ ($\lambda = 0.1$) and for $\tau \leq 50$, respectively. For later changes, the HWMA chart features astonishingly small D_τ values. However, this is not surprising because the HWMA control chart behaves like a Shewhart chart if the change happens quite late. Therefore, we added the Shewhart ARL values (deploying $c_H = 2.938$ as control limit of a Shewhart chart) to Figure 6. We conclude that for very large late changes, the HWMA chart performs well, but in these cases the Shewhart chart performs even better. In conclusion, we see no advantages whatsoever in using the HWMA approach, only disadvantages.

3.3 | Other HWMA research

Since the first paper on HWMA by Abbas,[22] there has been a rapid growth in research to investigate other versions of the HWMA method. One can refer to Adegoke et al[60] (multivariate HWMA), Adeoti and Koleoso[61] (HWMA for the process mean), Alevizakos et al[62] (double HWMA, or DWMA), Abid et al[63] (DWMA control chart for the process mean), Raza et al[64] (nonparametric HWMA), Riaz et al[65] (nonparametric DWMA), Abid et al[66] (mixed HWMA-CUSUM control chart), Abid et al[67] (mixed HWMA-CUSUM control chart for the process mean), Thanwane et al[68–71] (the effect of measurement error on HWMA), and Riaz et al[72] (triple HWMA or TWMA).

In particular, we note that the double and triple HWMA charts are simply reparameterizations of the HWMA chart. The consequences of this equivalency have not been recognized in the literature.

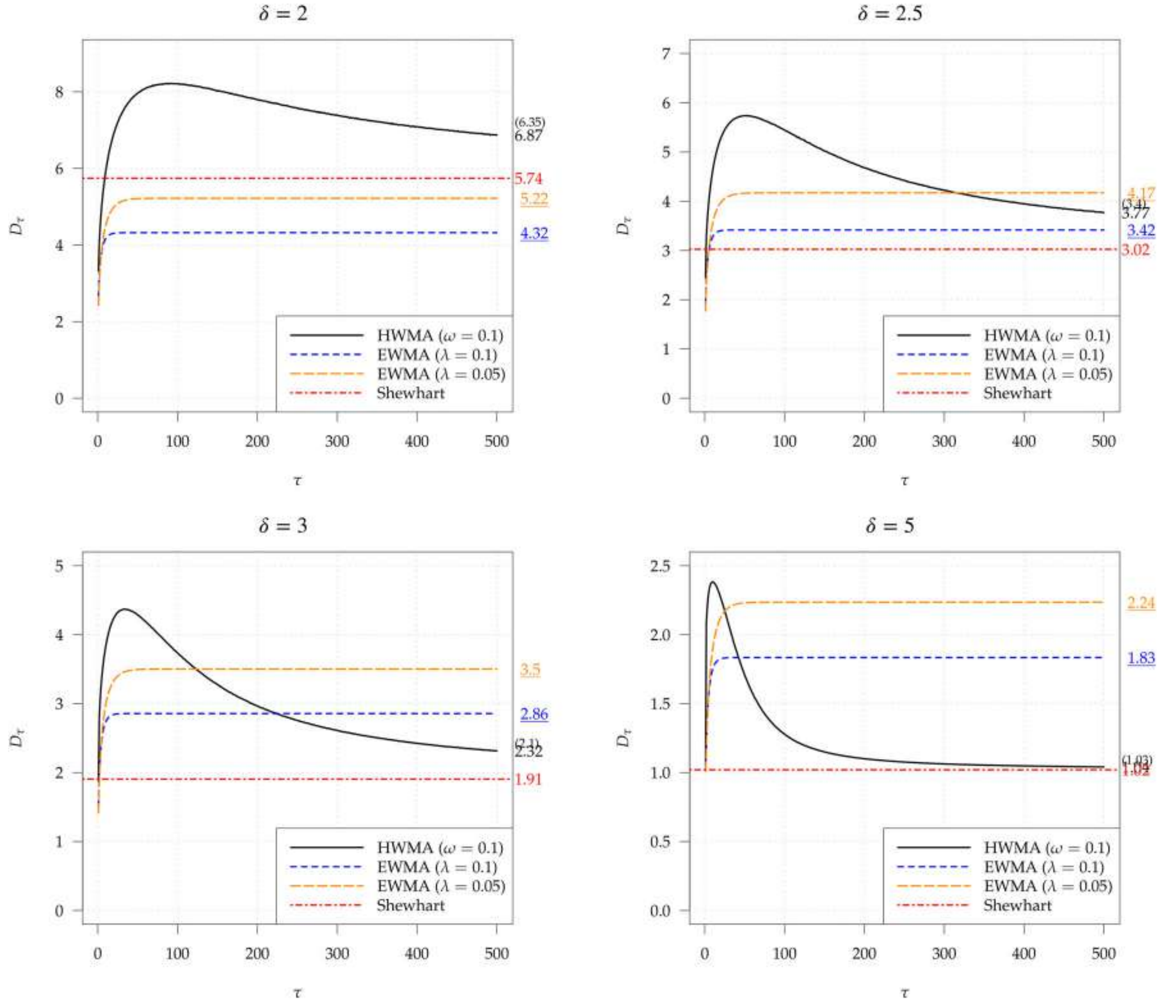


FIGURE 6 CED D_τ ($\delta \in \{2, 2.5, 3, 5\}$), Monte Carlo with 10^6 replicates, HWMA chart, two EWMA charts

4 | WEIGHTING PATTERNS

Following Lai's[73] statement, *"In utilizing previous observations for the detection of the lack of control, we have to ensure that the 'good old days' of the machine would not outweigh its present misery,"* one should never use weighting patterns such as the ones we observe for PM, HWMA, and some other recently proposed memory control charts. Note that the EWMA chart, the CUSUM chart, and even the simple moving average (MA) chart follow Lai's[73] design rules for the general formulation $S_t = \sum_{i=1}^{t} c_{t-i} X_i$ with $c_0 \ge c_1 \ge c_2 \ge \cdots c_{k-1} > 0 = c_k = c_{k+1} = \cdots$ The EWMA weights $c_i = \lambda(1-\lambda)^i$ are strictly monotonically decreasing with $c_i \approx 0$ for $i \gg 1$. The weights for the one-sided CUSUM chart are $c_0 = c_1 = \cdots c_{K-1} > 0$, with K being the random number of CUSUM observations after the last CUSUM statistic value of zero. For the MA chart, the parameter K is the window size. All other weights vanish, that is, $0 = c_K = c_{K+1} = \cdots$.

Now, we want to compare the weights of the PM chart, the DPM chart introduced in Riaz et al,[48] and the HWMA chart with the weights of the EWMA chart, the DEWMA chart from Shamma et al[74] and more recently in Zhang et al,[75] and the TEWMA chart proposed in Alevizakos et al.[76] Next we provide illustrations for $t = 2, 5, 10$, and 100. In Figure 7, we present the weight profiles for $t = 2$ and $= 5$. The first distinctive feature on the right-hand side of the figure is the heavy weight for the initial value (usually the in-control mean μ_0) of all EWMA schemes, which is more pronounced for the more complicated DEWMA and TEWMA charts.[77] In addition, proceeding in time, these large weights converge much more slowly to zero than for the EWMA chart. The other weights of the EWMA chart are strictly increasing with observation number (decreasing with age), whereas the weights for the DEWMA and TEWMA charts exhibit local maxima (easy to see for $t = 100$ in Figure 8). Thus, the DEWMA and TEWMA approaches violate Lai's weighting rules, and common sense, substantially.

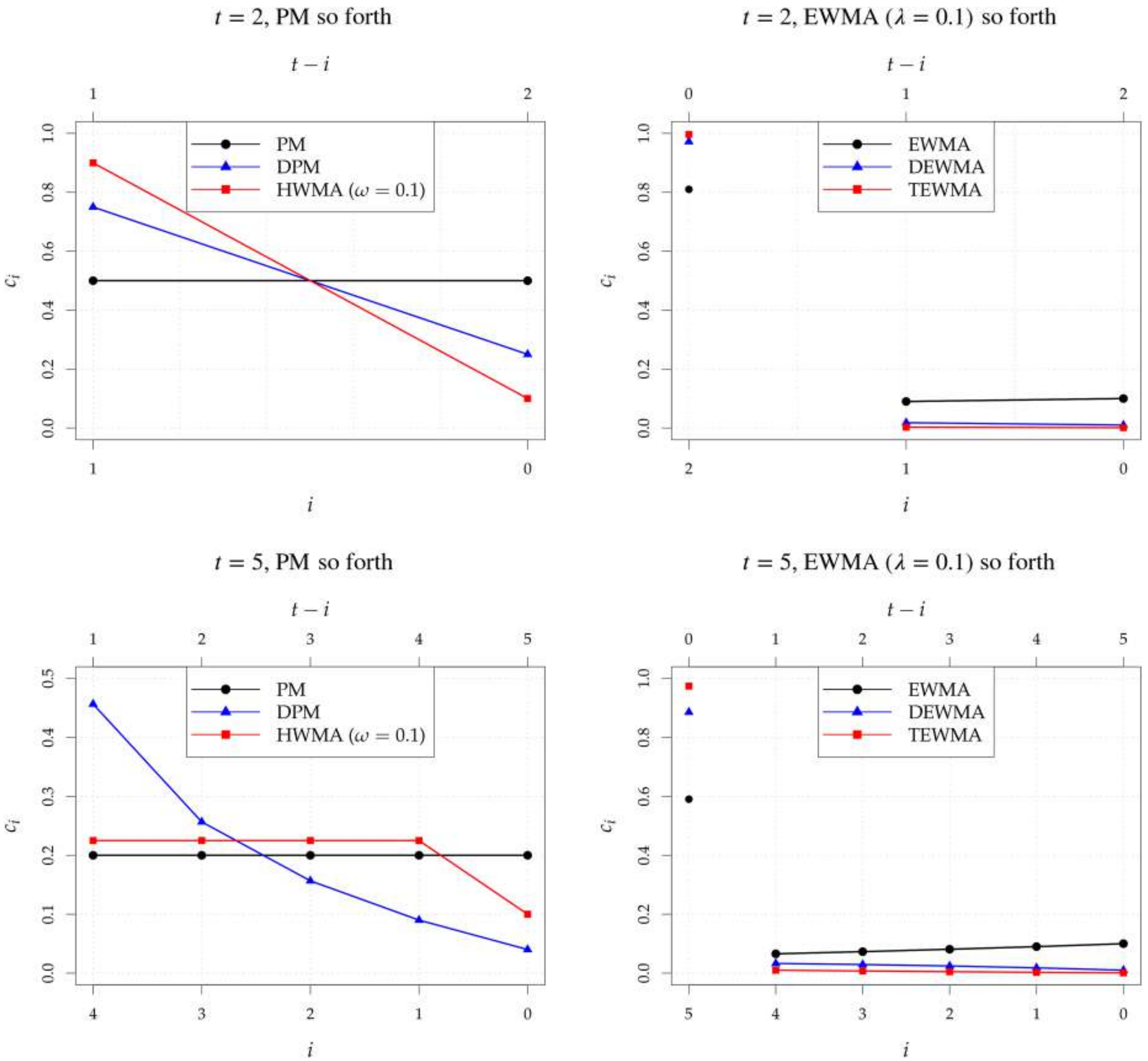


FIGURE 7 Weights c_i (Lai[73] notation) attached to all observations at time $t \in \{2, 5\}$

The most simple weighting pattern can be observed for the PM chart, where all observations get the same weight, $1/t$. The more recent HWMA features for $t < 10 = 1/\omega$ the lowest weight (ω) at t with the other weights being constant at $(1-\omega)/t$. For $t = 10$, the PM and HWMA weights coincide. Starting with $t > 10$, the HWMA chart gives the largest weight to the most recent observation (see Figure 8 for $t = 100$), whereas the other weights are all equal and small. This explains the similarity in detection behavior to that of a Shewhart chart when large changes happen late. The most counterintuitive and inappropriate weighting can be seen for the DPM chart, where the Lai rule is reversed. The most recent data are weighted less than data that were observed in the past. Here, the control chart design puts much weight on the "good old days," making it nearly impossible to detect changes that occur later than at start-up.

Aiming at excellent zero-state ARL performance was already considered by Chandrasekaran et al,[14] Rhoads et al,[78] and Steiner,[79] where the original EWMA design was reshaped to downsize the out-of-control zero-state ARL for certain changes. However, with Knoth,[15] it became clear that already the original setup with control limits utilizing the exact EWMA variance is an appropriate method to improve the detection performance for initial changes. Here, we want to illustrate its simple deployment to beat all the new methods by lowering the constant smoothing value λ. To make this easy, we determined the maximum λ to attain an out-of-control zero-state ARL not larger than the PM and HWMA values, respectively. The PM values in Figure 9 tell us that for medium and large δ, nearly all EWMA designs exhibit smaller out-of-control zero-state ARL values. For small $\delta < 1.2$, one has to pick $0.008 \le \lambda < 0.1$, and then it results on a uniformly quicker EWMA chart. For the HWMA chart with $\omega = 0.1$, we identify two patterns. First, for large $\delta > 4$ the bound $\lambda_{\max}$ decreases. Second, for all other δ not too small ($\delta > 0.5$), the standard $\lambda = 0.1$ defines an EWMA control chart that is

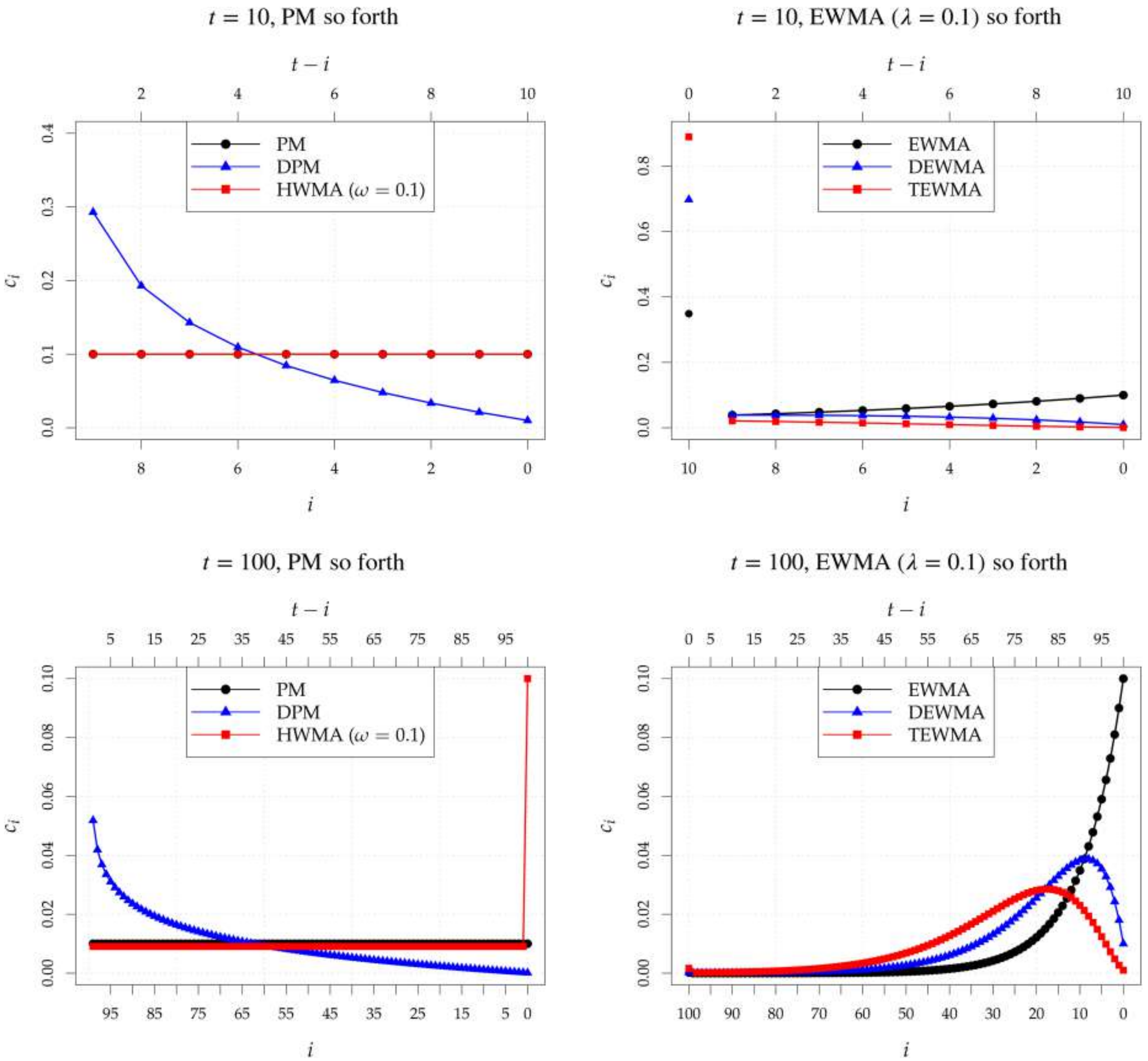


FIGURE 8 Weights c_i (Lai[73] notation) attached to all observations at time $t \in \{10, 100\}$

better in terms of both ARL types for a wide range of changes. One must remember that the zero-state comparison is only a small part of the performance evaluation. Taking very small λ allows the EWMA chart to beat the PM or HWMA charts, but these very special setups lead to steady-state ARL problems similar to those new charts. To give an idea, we consider $\lambda \in (0.001, 1)$ for an in-control ARL $E_\infty(L_E) = 500$ and determine both the zero-state and steady-state ARL for selected changes. From Figure 10A, we conclude that the out-of-control zero-state ARL decreases with decreasing λ. Hence, in a simple zero-state ARL competition, one utilizes a very small λ. From Tables 1 and 2, we know that these λ are not this small ($\lambda = 0.007$ to beat PM, $\lambda = 0.05$ to win against the HWMA chart). However, in Figure 10B, we recognize that the steady-state ARL $\mathcal{D}_\delta$ profiles feature local minima, whose arguments are quite large for large δ. We should recall that for constant control limits relying on the asymptotic EWMA variance $\lambda/(2-\lambda)$, the zero-state ARL profiles exhibit a similar shape as the $\mathcal{D}_\delta$ curves.

5 | CONCLUSIONS

We showed that the PM and HWMA approaches should never be used in practice. It is important in SPM to give less weight to data which move further and further into the past. These two approaches, in general, do not possess this important property, resulting in very poor performance in detecting delayed shifts.

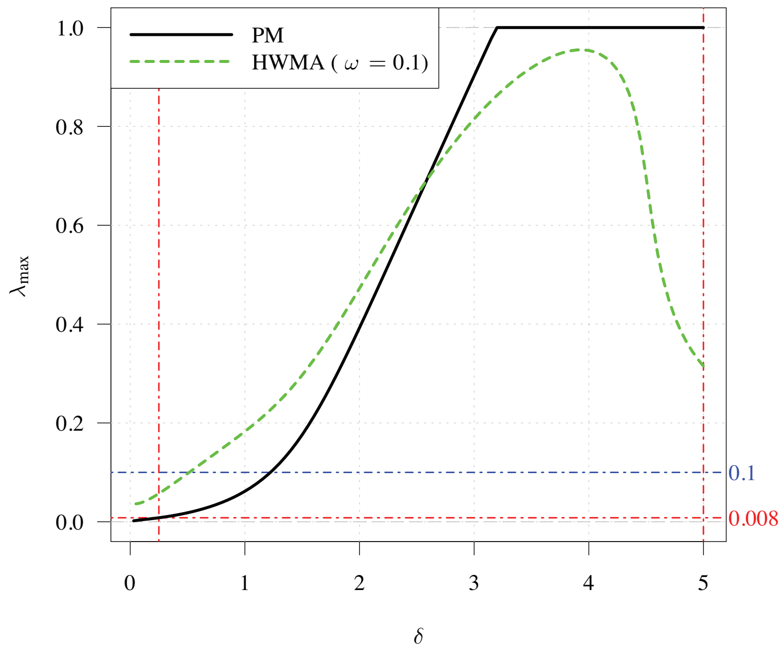


FIGURE 9 Maximum λ value to "beat" the zero-state ARL of PM and HWMA, $E_\infty(L_E) = 500$

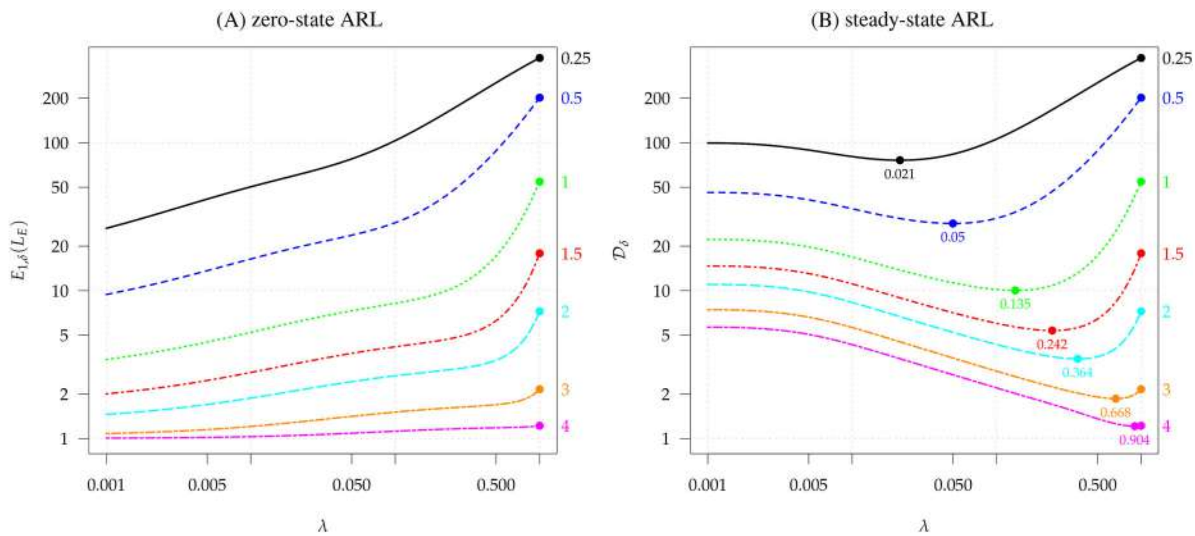


FIGURE 10 ARL performance for EWMA, revisited, $E_\infty(L_E) = 500$, shifts $\delta \in \{0.25, 0.5, \ldots, 4\}$

We believe that these schemes that follow the PM and HWMA principles are more closely related to sequential hypothesis testing, in the sense of Wald[80] or Dodge,[81] than to process monitoring approaches appropriate for quality management practice. Sequential hypothesis testing and control charts are related, and, sometimes, they overlap. However, they are based on different objectives, and they are applied in different circumstances. Sequential hypothesis testing evaluates evidence as it is incorporated until a conclusion is reached in a situation where the population is stable, and the practitioner wants to conclude with a minimum cost. A control chart can also be used to assess a population sequentially, but they were not designed for a stable population. Process monitoring schemes are built to keep a watch over a process with enough sensitivity to detect isolated or sustained changes if and when they occur. A new process might undergo different behavior from expectation the moment it starts; however, as a system evolves, as it gets better, more capable, it also becomes more stable, to a point where hundreds or thousands of measurements might flow under control until an unexpected assignable cause results in a large enough change that matters. A proper chart design, to be useful in process monitoring, must consider the change point to be unknown (either deterministic or stochastic), where steady-state performance and CED performance play a role even more fundamental than zero-state behavior.

As a final note, it is implicitly assumed in our paper, and all papers on the PM and HWMA approaches, that any process change, however small, is to be detected quickly. If some process shifts are considered too small to be of concern, then we recommend the approach of Woodall and Faltin.[82]

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