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# SOME LOCALLY PROJECTIVELY FLAT WITH $(\alpha, \beta)$ METRIC

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This paper has been devoted to the study of Some locally projectively flat with $(\alpha, \beta)$ matric. In this paper first section is introductory. In the second section we have studied locally projectively flat with- $(\alpha, \beta)$ metric. In this section we have studied some related definitions on projective change and locally projectively flat and using the relation: $F = \frac{\alpha^2}{\beta}$ (V.K. Kropina [9]), (called Kropina metric), $F = \frac{\alpha^2}{\alpha - \beta}$ (*M.* Matsumoto [10]), and $F = \alpha + \frac{\beta^2}{\alpha}$ (M. Matsumoto [11]) where $\alpha$ is a Riemannian matric and $\beta$ is a 1-form defined on the *n*-dimensional differentiable manifold M. In the light of these observations we get some results when $F$ and $\alpha$ be locally projectively flat.

**Keywords:** Locally projectively flat, Projective change.

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## INTRODUCTION

The projective changes between two Finsler space with $(\alpha, \beta)$- matric have been studied by S. Bacso and M. Matsumoto [7], Hong Park and Yong Lee [8], and M. Matsumoto [4], M. Hashiguchi and Y. Icijyo [5], C. Shibata [6] studied the projective changes between a Finsler space with $(\alpha, \beta)$- matric and its associated Riemannian space.

M. Matsumoto [1] introduced the concept of $(\alpha, \beta)$- matric on a differentiable manifold. $\alpha^2 = \alpha_{ij}(x) y^i y^j$ and $\beta = b_i(x) y^i$, where $\alpha$ is a Riemannian matric and $\beta$ is a 1-form defined on the *n*-dimensional differentiable manifold M. Now we use the following forms was introduced by:

$$F = \frac{\alpha^2}{\beta} \qquad \text{(V.K. Kropina [9]), (called Kropina metric)} \qquad ...(1.1)$$

$$F = \frac{\alpha^2}{\alpha - \beta} \quad \text{(M. Matsumoto [10])}, \qquad ...(1.2)$$

$$F = \alpha + \frac{\beta^2}{\alpha} \quad \text{(M. Matsumoto [11])}. \qquad ...(1.3)$$

Here, $F_n = (M, F)$ is said to be Finsler space with $(\alpha, \beta)$- metric.

## LOCALLY PROJECTIVELY FLAT WITH $(\alpha, \beta)$- METRIC:

We now give the following definitions which will be use in the later discussions:

### Definition (1.1):

Let us consider a transformation $F_n = (M, F)$ *to* $\overline{F_n} = (M, \bar{F})$ between two Finsler spaces $F_n$ and $\overline{F_n}$ defined over the same underlying manifold $M$ of dimension $n$, which is a diffeomorphism and maps geodesics of $F_n$ to geodesics of $\overline{F_n}$. Such type of transformation is called a projective transformation or projective change.

A Finsler space $F_n$ is projective to another Finsler space $\overline{F_n}$ iff there exists a $P(x, \dot{x})$ is positively homogeneous scalar function of degree one in direction $\dot{x}^i (= y^i)$, such that

$$\bar{G}^i(x, \dot{x}) = G^i(x, \dot{x}) + P(x, \dot{x})\dot{x}^i , \qquad ...(2.1)$$

the scaler field $P = P(x, \dot{x})$ is called the projective factor of the projective change (Z.Shen [13]).

A. Rapcsak [2] has been defined the following set of equation

$$\bar{G}^i(x, \dot{x}) = G^i(x, \dot{x}) + \frac{\bar{F}_{|k} y^k}{2\bar{F}} y^i + \frac{\bar{F}}{2} g^{-il} \left\{ \frac{\partial \bar{F}_{|k}}{\partial y^l} y^k - \bar{F}_{|l} \right\} \qquad ...(2.2)$$

where $\bar{F}_{|k}$ denotes the horizontal covariant derivative of $\bar{F}$ on $F_n = (M, F)$ given by

$$\bar{F}_{|k} = \frac{\partial \bar{F}}{\partial x^k} - \frac{\partial G^r}{\partial y^k} \frac{\partial \bar{F}}{\partial y^r} \qquad ...(2.3)$$

Let $F_n = (M, F)$ and $\overline{F_n} = (M, \bar{F})$ be two Finsler spaces defined on same underlying manifold $M$ of dimension $n$. The change $F \to \bar{F}$ of the matric is a projective change iff $\bar{F}$ satisfies

$$\bar{F}_{|i} - \frac{\partial \bar{F}_{|k}}{\partial y^i} y^k = 0 . \qquad ...(2.4)$$

Here, $F_n$ projective to $\overline{F_n}$ with the projective factor given by

$$P = \frac{\bar{F}_{|k} y^k}{2\bar{F}} \qquad ... (2.5)$$

If projective change $F \rightarrow \bar{F}$ of a Finsler space $F_n = (M, F)$ such that the Finsler space $\bar{F_n} = (M, \bar{F})$ is a locally Minkowski space then $F_n$ is called locally projective flat.

If $\bar{F_n}$ is a locally Minkowski space then $\bar{G}^i(x, \dot{x}) = 0$, therefore the set of equation (2.2) becomes

$$G^i(x, \dot{x}) = -P(x, \dot{x})\,\dot{x}^i \qquad \text{...(2.6)}$$

where $P$ is the projective factor of the projective change $F \rightarrow \bar{F}$.

Now we use the following lemma which gives the requirement for any Finsler metric to be locally projectively flat (G. Hamel [3]).

**Lemma (2.1):**

A Finsler space $\boldsymbol{F_n = (M, F)}$ is locally projectively flat iff

$$\boldsymbol{\frac{\partial \bar{F}}{\partial x^i} - \frac{\partial^2 F}{\partial x^k \partial y^i} y^k = 0.} \qquad \text{... (2.7)}$$

We now obtain the projective factor $\bar{P}$ of the projective change $F \rightarrow \bar{F}$, where $F$ is locally projectively flat, the set of equation (2.1) becomes

$$G^i(x, \dot{x}) = \bar{G}^i(x, \dot{x}) + \bar{P}(x, \dot{x})\dot{x}^i \qquad \text{...(2.8)}$$

Since, $\bar{G}^i(x, \dot{x}) = 0$, therefore we get

$$G^i(x, \dot{x}) = \bar{P}(x, \dot{x})\dot{x}^i \qquad \text{...(2.9)}$$

The set of equation (2.5) and $\bar{F}_{|k} = \frac{\partial F}{\partial x^k}$, we have

$$\bar{P} = \frac{1}{2F}\frac{\partial F}{\partial x^k}\dot{x}^k \qquad \text{... (2.10)}$$

The set of equation (2.6) and (2.9) becomes

$$\bar{P} = -P \qquad \text{...(2.11)}$$

Now Let us suppose that $F$ is Locally projectively flat Kropina metric (1.1). The set of equation (2.7) becomes

$$\frac{\partial}{\partial y^i}\left(\frac{\partial F}{\partial x^k}\right) y^k - \frac{\partial \bar{F}}{\partial x^i} = 0, \qquad \text{... (2.12)}$$

where, $$\frac{\partial F}{\partial x^k} = \frac{\partial}{\partial x^k}\left(\frac{\alpha^2}{\beta}\right) = \frac{2\alpha}{\beta}\frac{\partial \alpha}{\partial x^k} - \frac{\alpha^2}{\beta^2}\frac{\partial \beta}{\partial x^k}. \qquad \text{... (2.13)}$$

Differentiating (2.13) with respect to $y^i$ and contracting with $y^k$, we get

$$\frac{\partial}{\partial y^i}\left(\frac{\partial F}{\partial x^k}\right) y^k = 2\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right)\frac{\partial \alpha}{\partial x^k} y^k + \frac{2\alpha}{\beta}\frac{\partial}{\partial y^i}\left(\frac{\partial \alpha}{\partial x^k}\right) y^k - \frac{2\alpha}{\beta}\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right)\frac{\partial \beta}{\partial x^k} y^k - \\ - \frac{\alpha^2}{\beta^2}\frac{\partial b_i}{\partial x^k} y^k, \qquad \text{... (2.14)}$$

where, $\beta = b_i(x)y^i$.

Now, Replacing $k$ by $i$ in (2.13) and using $\beta = b_i(x)y^i$, we have

$$\frac{\partial F}{\partial x^i} = \frac{2\alpha}{\beta}\frac{\partial \alpha}{\partial x^i} - \frac{\alpha^2}{\beta^2}\frac{\partial b_k}{\partial x^i}y^k \quad \text{... (2.15)}$$

We now using the set of equation (2.12), (2.14) and (2.15), we have

$$2\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right)\left\{\frac{\partial \alpha}{\partial x^k} - \frac{\alpha}{\beta}\frac{\partial \beta}{\partial x^k}\right\}y^k + \frac{2\alpha}{\beta}\left\{\frac{\partial}{\partial y^i}\left(\frac{\partial \alpha}{\partial x^k}\right)y^k - \frac{\partial \alpha}{\partial x^i}\right\}$$
$$+\frac{\alpha^2}{\beta^2}\left(\frac{\partial b_k}{\partial x^i} - \frac{\partial b_i}{\partial x^k}\right)y^k = 0 \quad \text{... (2.16)}$$

In the light of these observations, therefore we can state:

### Theorem (2.1):

**In a Finsler space $F_n$, If $F$ be locally projectively flat then the relation (2.16) holds (where $F$ be a Kropina metric (1.1), $\alpha$ is a Riemannian matric and $\beta$ is a 1-form defined on an $n$-dimensional differentiable manifold M). The converse of this theorem is also true.**

Now since $\alpha$ is locally projectively flat, then the set of equation (2.7) become

$$\frac{\partial}{\partial y^i}\left(\frac{\partial \alpha}{\partial x^k}\right)y^k - \frac{\partial \alpha}{\partial x^i} = 0, \quad \text{... (2.17)}$$

and the set of equation (2.10), we can get

$$P = \frac{1}{2\alpha}\frac{\partial\, \alpha}{\partial x^k}y^k. \quad \text{... (2.18)}$$

Using the set of equation (2.16) and (2.17), we have

$$2\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right)\left\{\frac{\partial \alpha}{\partial x^k} - \frac{\alpha}{\beta}\frac{\partial \beta}{\partial x^k}\right\}y^k = -\frac{\alpha^2}{\beta^2}\left(\frac{\partial b_k}{\partial x^i} - \frac{\partial b_i}{\partial x^k}\right)y^k. \quad \text{... (2.19)}$$

The set of equation (2.19) can be also written as

$$4\alpha\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\beta}\right)(\mathrm{P} - \mathrm{Q}) = -\frac{\alpha^2}{\beta^2}\left(\frac{\partial b_k}{\partial x^i} - \frac{\partial b_i}{\partial x^k}\right)y^k, \quad \text{... (2.20)}$$

where $P = \frac{1}{2\alpha}\frac{\partial\, \alpha}{\partial x^k}y^k$ and $Q = \frac{1}{2\beta}\frac{\partial\, \beta}{\partial x^k}y^k$

In the light of these observations, therefore we can state:

### Theorem (2.2):

**In a Finsler space $F_n$, If $F$ be locally projectively flat and $\alpha$ is locally projectively flat then the relation (2.20) holds (where $F$ be a Kropina metric).**

Now Let us suppose that $F$ is Locally projectively flat and using (1.2). Then the set of equation (2.7) becomes

$$\frac{\partial}{\partial y^i}\left(\frac{\partial F}{\partial x^k}\right)y^k - \frac{\partial \bar{F}}{\partial x^i} = 0, \qquad \ldots (2.21)$$

where,

$$\frac{\partial F}{\partial x^k} = \frac{\partial}{\partial x^k}\left(\frac{\alpha^2}{\alpha-\beta}\right) = \frac{2\alpha}{(\alpha-\beta)}\frac{\partial \alpha}{\partial x^k} - \frac{\alpha^2}{(\alpha-\beta)^2}\left(\frac{\partial \alpha}{\partial x^k} - \frac{\partial \beta}{\partial x^k}\right). \qquad \ldots (2.22)$$

Differentiating (2.22) with respect to $y^i$ and contracting with $y^k$, we get

$$\frac{\partial}{\partial y^i}\left(\frac{\partial F}{\partial x^k}\right)y^k = 2\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\alpha-\beta}\right)\left\{\frac{\partial \alpha}{\partial x^k} - \frac{\alpha}{\alpha-\beta}\frac{\partial \alpha}{\partial x^k}\right\}y^k +$$

$$\frac{2\alpha}{\alpha-\beta}\left\{\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\alpha-\beta}\right)\frac{\partial \beta}{\partial x^k} + \quad + \frac{\partial}{\partial y^i}\left(\frac{\partial \alpha}{\partial x^k}\right)\right\}y^k + \frac{\alpha^2}{(\alpha-\beta)^2}\left\{\frac{\partial b_i}{\partial x^k} - \frac{\partial}{\partial y^i}\left(\frac{\partial \alpha}{\partial x^k}\right)\right\}y^k, \qquad \ldots (2.23)$$

Now, replacing $k$ by $i$ in the set of equation (2.22), we get

$$\frac{\partial F}{\partial x^i} = \frac{2\alpha}{(\alpha-\beta)}\frac{\partial \alpha}{\partial x^i} - \frac{\alpha^2}{(\alpha-\beta)^2}\left(\frac{\partial \alpha}{\partial x^i} - \frac{\partial \beta}{\partial x^i}\right). \qquad \ldots (2.24)$$

Thus, using the set of equation (2.22) and (2.24) in (2.21), we get

$$\frac{2\beta}{\alpha-\beta}\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\alpha-\beta}\right)\left\{\frac{\alpha}{\beta}\frac{\partial \beta}{\partial x^k} - \frac{\partial \alpha}{\partial x^k}\right\}y^k + \frac{\alpha^2}{(\alpha-\beta)^2}\left\{\frac{\partial b_i}{\partial x^k} - \frac{\partial b_k}{\partial x^i}\right\}y^k +$$

$$+\frac{(\alpha^2-2\alpha\beta)}{(\alpha-\beta)^2}\left\{\frac{\partial}{\partial y^i}\left(\frac{\partial \alpha}{\partial x^k}\right)y^k - \frac{\partial \alpha}{\partial x^i}\right\} = 0. \qquad \ldots (2.25)$$

In the light of these observations, therefore we can state:

**Theorem (2.3):**

**In a Finsler space $F_n$, If $F$ be locally projectively flat then the relation (2.25) holds $\left(\text{where } F = \dfrac{\alpha^2}{\alpha-\beta}\right)$, $\alpha$ is a Riemannian matric and $\beta$ is a 1-form defined on an *n*-dimensional differentiable manifold M). The converse of this theorem is also true.**

Now since $\alpha$ is locally projectively flat, then the set of equation (2.25) become

$$\frac{2\beta}{\alpha-\beta}\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\alpha-\beta}\right)\left\{\frac{\alpha}{\beta}\frac{\partial \beta}{\partial x^k} - \frac{\partial \alpha}{\partial x^k}\right\}y^k + \frac{\alpha^2}{(\alpha-\beta)^2}\left\{\frac{\partial b_i}{\partial x^k} - \frac{\partial b_k}{\partial x^i}\right\}y^k = 0 \qquad \ldots (2.26)$$

The set of equation (2.26) can be also written as

$$\frac{\partial}{\partial y^i}\left(\frac{\alpha}{\alpha-\beta}\right)(P-Q) = -\frac{1}{4\beta}\frac{\alpha}{(\alpha-\beta)}\left(\frac{\partial b_k}{\partial x^i} - \frac{\partial b_i}{\partial x^k}\right)y^k, \qquad \ldots (2.27)$$

where $P = \frac{1}{2\alpha}\frac{\partial \alpha}{\partial x^k} y^k$ and $Q = \frac{1}{2\beta}\frac{\partial \beta}{\partial x^k} y^k$

In the light of these observations, therefore we can state:

**Theorem (2.4):**

In a Finsler space $\boldsymbol{F_n}$ , If $\boldsymbol{F}$ be locally projectively flat and $\boldsymbol{\alpha}$ is locally projectively flat then the relation (2.27) holds $\left(\text{where}\, F = \frac{\alpha^2}{\alpha - \beta}\right)$.

Now Let us suppose that $F$ is Locally projectively flat and using (1.3). Then the set of equation (2.7) becomes

$$\frac{\partial}{\partial y^i}\left(\frac{\partial F}{\partial x^k}\right) y^k - \frac{\partial \bar{F}}{\partial x^i} = 0, \qquad \text{... (2.28)}$$

where,

$$\frac{\partial F}{\partial x^k} = \frac{\partial}{\partial x^k}\left(\alpha + \frac{\beta^2}{\alpha}\right) = \frac{\partial \alpha}{\partial x^k} + \frac{2\beta}{\alpha}\frac{\partial \beta}{\partial x^k} - \frac{\beta^2}{\alpha^2}\frac{\partial \alpha}{\partial x^k}. \qquad \text{... (2.29)}$$

Differentiating (2.29) with respect to $y^i$ and contracting with $y^k$, we get

$$\frac{\partial}{\partial y^i}\left(\frac{\partial F}{\partial x^k}\right) y^k = 2\, \frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)\left\{\frac{\partial \beta}{\partial x^k} - \frac{\beta}{\alpha}\frac{\partial \alpha}{\partial x^k}\right\} y^k$$
$$+ \frac{\alpha^2 - \beta^2}{\alpha^2}\, \frac{\partial}{\partial y^i}\left(\frac{\partial \alpha}{\partial x^k}\right) y^k + \frac{2\beta}{\alpha}\frac{\partial b_i}{\partial x^k} y^k, \qquad \text{... (2.30)}$$

Now, replacing $k$ by $i$ in the set of equation (2.29), we get

$$\frac{\partial F}{\partial x^i} = \frac{\partial \alpha}{\partial x^i} + \frac{2\beta}{\alpha}\frac{\partial \beta}{\partial x^i} - \frac{\beta^2}{\alpha^2}\frac{\partial \alpha}{\partial x^i}. \qquad \text{... (2.31)}$$

Thus, using the set of equation (2.30) and (2.31) in (2.28), we get

$$2\, \frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)\left\{\frac{\partial \beta}{\partial x^k} - \frac{\beta}{\alpha}\frac{\partial \alpha}{\partial x^k}\right\} y^k + \frac{2\beta}{\alpha}\left(\frac{\partial b_i}{\partial x^k} - \frac{\partial b_k}{\partial x^i}\right) y^k$$
$$+ \frac{\alpha^2 - \beta^2}{\alpha^2}\left\{\frac{\partial}{\partial y^i}\left(\frac{\partial \alpha}{\partial x^k}\right) y^k - \frac{\partial \alpha}{\partial x^i}\right\} = 0 \qquad \text{... (2.32)}$$

In the light of these observations, therefore we can state:

**Theorem (2.5):**

**In a Finsler space $\boldsymbol{F_n}$ , If $\boldsymbol{F}$ be locally projectively flat then the relation (2.32) holds (where $\boldsymbol{F = \alpha + \frac{\beta^2}{\alpha}}$ , $\boldsymbol{\alpha}$ is a Riemannian matric and $\boldsymbol{\beta}$ is a 1-form defined on an *n*-dimensional differentiable manifold M). The converse of this theorem is also true.**

Now since $\alpha$ is locally projectively flat, then the set of equation (2.32) become

$$\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)\left\{\frac{\partial \beta}{\partial x^k}-\frac{\beta}{\alpha}\frac{\partial \alpha}{\partial x^k}\right\}y^k+\frac{\beta}{\alpha}\left(\frac{\partial b_i}{\partial x^k}-\frac{\partial b_k}{\partial x^i}\right)y^k=0 \quad \ldots(2.33)$$

Now the set of equation (2.33) can be also written as

$$\frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right)(P-Q)=-\frac{1}{2\alpha}\left(\frac{\partial b_k}{\partial x^i}-\frac{\partial b_i}{\partial x^k}\right)y^k, \quad \ldots(2.34)$$

where $P=\frac{1}{2\alpha}\frac{\partial \alpha}{\partial x^k}y^k$ and $Q=\frac{1}{2\beta}\frac{\partial \beta}{\partial x^k}y^k$

In the light of these observations, therefore we can state:

**Theorem (2.6):**

**In a Finsler space $F_n$ , If $F$ be locally projectively flat and $\alpha$ is locally projectively flat then the relation (2.34) holds** $\left(\text{where } F=\alpha+\frac{\beta^2}{\beta}\right)$.

## CONCLUSION

This paper is divided into two sections of which the first section is introductory. In the second section we have studied locally projectively flat with- $(\alpha,\beta)$ metric. In this section we have studied some related definitions on projective change and locally projectively flat. we use the lemma (2.1) which gives the requirement for any Finsler metric to be locally projectively flat. Using the following relations: $F=\frac{\alpha^2}{\beta}$, $F=\frac{\alpha^2}{\alpha-\beta}$ and $F=\alpha+\frac{\beta^2}{\alpha}$ where $\alpha$ is a Riemannian matric and $\beta$ is a 1-form defined on the $n$-dimensional differentiable manifold M. In the light of these observations when $F$ and $\alpha$ be locally projectively flat we get some results in the form of theorems.

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