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F. Zandi[a], S.T.A. Niaki[b]*, M.A. Nayeri[c] and M. Fathi[d]

[a]*Department of Industrial Engineering, K.N. Toosi University of Technology, Tehran, Iran;* [b]*Department of Industrial Engineering, Sharif University of Technology, Tehran, Iran;* [c]*Department of Industrial Engineering, Amirkabir University of Technology, Tehran, Iran;* [d]*Department of Industrial Engineering, Iran University of Science and Technology, Tehran, Iran*

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Despite the fact that control charts are able to trigger a signal when a process has changed, it does not indicate when the process change has begun. The time difference between the changing point and a signal of a control chart could cause confusions on the sources of the problems. Knowing the exact time of a process change would help to reduce the time for identification of the special cause. In this article, a model for the change-point problem is first introduced and a maximum-likelihood estimator (MLE) is applied when a linear trend disturbance is present. Then, Monte Carlo simulation is applied in order to evaluate the accuracy and the precision performances of the proposed change-point estimator. Next, the proposed estimator is compared with the MLE of the process fraction non-conforming change point derived under simple step and monotonic changes following signals from a Shewhart *np* control chart. The results show that the MLE of the process change point designed for the linear trend outperforms the MLE designed for step and monotonic changes when a linear trend disturbance is present.

Keywords: change point; process fraction non-conforming; statistical process control; process improvement; *np* charts; maximum-likelihood estimation

1. Introduction and literature review

Statistical process control (SPC) has played an important role in industry for many years. The control chart is a powerful SPC tool that monitors the changes and discovers variation in a process in order to distinguish between special and common causes of variation. In SPC, upper and lower control limits can be defined based on the probability distribution of the product's quality characteristics.

When the sample observations of the process are placed within the control limits, it can be concluded that the process is in control. However, if the sample observations are placed outside the control limits, an out-of-control signal is received. When a control chart signals an out-of-control condition, a search begins to identify and eliminate the source(s) of the special cause (see Montgomery 1996 for more details on control charts). 'The time when a special cause manifests itself into a process is referred to as change point' (Atashgar and Noorossana 2010).

Control chart's signal shows that process engineers can begin their search for the special cause of change in the process. Moreover, the disturbance in a process can be accomplished from special causes or common causes. Although control charts suggest occurrence of a change, neither can they show specific information on the cause of process disturbance, nor do they show the time of the process disturbance. In the literature of control charting methods, the change point is the time when a process begins its change by a single or multiple disturbances. However, the signalling time is the time when a control chart signals the existence of an assignable cause. Knowing the exact point of change in a process would help to search and identify special causes, resulting in time saving to find the causes. Therefore, it is useful to identify the difference between the change point and the time when an out-of-control signal is generated by control charts (Bassevile and Nikiforov 1993).

Industrial quality control setting often uses the binomial distribution to model the number of defective items in a sample of size n. Process fraction non-conforming, p, is the probability that a randomly selected item does not conform the quality characteristic. That is, given n items, the probability that x randomly selected items is defective is given by

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x};$$
$$x = 1, 2, \ldots, n \quad 0 \le p \le 1$$

*Corresponding author. Email: Niaki@sharif.edu

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where p denotes the process fraction non-conforming. Depending on whether the subgroup size is constant or not, one often uses p or np charts to monitor a process. Moreover, the np, the cumulative sum (CUSUM), and the exponentially weighted moving average (EWMA) control charts are commonly used to monitor binomial counts (Ryan 2000).

Recent literatures on change-point estimation are as follows:

Hawkins and Qiu (2003) studied the change-point model for SPC. Samuel et al. (1998a, 1998b) considered step change in a normal process mean and normal process variance. Pignatielo and Samuel (2001) proposed an estimator for the change point of a normal process mean, and, based on this study, Perry and Pignatiello (2006) proposed an maximum-likelihood estimator (MLE) and evaluated the performance of this estimator when a linear trend change is present in a normal process mean. They showed that their proposed estimator provides good performance when a linear trend disturbance is present. They compared their results with suggested estimator by Samuel et al. (1998a, 1998b) for step changes. Moreover, their results showed that the MLE obtained for linear trend disturbances outperforms the MLE obtained for step change disturbances in the presence of the linear trend disturbance. Samuel and Pignatiello (1998) analysed a step change in the rate parameter for a Poisson process. Nedumaran et al. (2000) addressed the issue of change-point identification for χ^2 control chart. They used MLE to estimate a step change shift in the mean of a normal distribution. Noorossana and Shademan (2009) proposed MLE for the change point of a normal process mean that does not require the knowledge of the exact change type showed by the process. The only required assumption is that the change type present should belong to a family of monotonic change, either isotonic or antitonic. Furthermore, they compared performances between their estimator and those suggested by Samuel et al. (1998a, 1998b) and Perry and Pignatiello (2006) following a genuine signal from the Shewhart $\overline{X}$ control chart. Noorossana et al. (2009) proposed an estimator for a period of time in which a step change in the process non-conformity proportion in high-yield processes occurs. Gazanfari et al. (2008) used clustering approach to identify the time of a step change in the Shewhart control charts.

Samuel and Pignatiello (2001) proposed a MLE for the process fraction non-conforming change point by applying the step change likelihood function. They evaluated the performances of their proposed estimator when a np chart signals and concluded that their estimator provides good accuracy and precision performances. Moreover, Perry et al. (2007) developed a change-point estimator from the change likelihood function for a binomial random variable without assuming the previous information of the exact change type. The only assumption in this research is that the predicted change type is belonging to a family of monotonic change type. Further, Perry et al. (2007) compared the performances between their estimator and the one suggested by Samuel and Pignatiello (2001). In this article, a MLE is proposed for the change point of the process fraction non-conforming using the change likelihood function for a linear trend disturbance. The proposed estimator can be used for the detection of a change point when either p or np chart has shown a signal. In their research, Monte Carlo simulation is used to evaluate performances of their estimator to the commonly used MLE for the time of step change and monotonic change when a linear trend disturbance is presented following a signal from a Shewhart np control chart.

In the current research work, the change-point problem of a process fraction non-conforming is first introduced and a MLE is applied when a linear trend disturbance is present. Examples of manufacturing processes in which special causes can happen due to the linear trend disturbances in fraction non-conforming involve gradual tool wear, machine depreciation, workers' fatigue, filters that become dirty over time, or any other time-related factors that can affect the quality of produced items. Manufacturing environments with high-quality products are also some examples, in which both the fraction non-conforming and its slope of change must be low. Then, Monte Carlo simulation is applied in order to evaluate the accuracy and the precision performances of the proposed change-point estimator. Next, the proposed estimator is compared with the MLE of the process fraction non-conforming change point derived under simple step and monotonic changes following the signals from a Shewhart np control chart.

The outline of this article is as follows. We study a model for disturbance in the process when a linear trend is present in Section 2. In Section 3, we evaluate and compare the precision and the accuracy performances of the estimator. Finally, we give some concluding remarks in Section 4.

2. Linear trend change model and MLE derivation

Consider a linear trend change model for the behaviour of a process fraction non-conforming p. It is assumed the process is initially in control for the first τ subgroups and independent observations are coming from a binomial distribution with in-control parameter $p = p_0$. Following an unknown point in time τ (the process change point), the first disturbance in the process fraction non-conforming happens. After this time, the process changes from $p = p_0$ to an out-of-control state p (where $p = p_i$; $i = \tau + 1, \tau + 2, \ldots, T$, and T denotes the time when a control chart generates a signal. A signal can be obtained when a point is either plotted above the upper control limit or an out-of-control pattern is detected using the Western Electric or other sensitising rules). Assuming the signal is not a false alarm, the change model of p is given by Equation (1), where β is the slope of the linear trend disturbance or the magnitude of process change.

$$p_i = p_0 + \beta(i - \tau) \tag{1}$$

In the proposed linear trend change model, each observation consists of a subgroup from the output of the process. For subgroups $i = 1, 2, \ldots, \tau$, the process is in control and the process fraction non-conforming is the known p_0. However, for subgroups $i = \tau + 1, \tau + 2, \ldots, T$, the process fraction non-conforming is some unknown $p_i = p_0 + \beta(i - \tau)$, where T is the most recent subgroup sample, i.e. the chart signals a change in p at subgroup number T. This model has two unknown parameters τ and β. The parameter τ represents the last subgroup taken from the in-control process, and β is the slope parameter of the linear trend model. The value of $\beta > 0$ denotes a linear change with an additive trend in p, while $\beta < 0$ represents a descending trend in the process fraction non-conforming. Based on these assumptions, the MLE can be derived for the process change point τ with non-decreasing change type ($\beta > 0$). The MLE change-point estimator is denoted by $\hat{\tau}_{lt}$.

Considering the model in Equation (1) and the above assumptions, and that the first change point takes place at time τ, the likelihood function becomes

$$L(\tau, \beta D) = \prod_{i=1}^{\tau} \binom{n}{x_i} p_0^{D_i}(1 - p_0)^{n-D_i} \prod_{i=\tau+1}^{\mathrm{T}} \binom{n}{x_i} \times p_i^{D_i}(1 - p_i)^{n-D_i} \tag{2}$$

where n is the size of the subgroup (the subgroups size is constant) and D_i denotes the number of non-conforming units in the ith subgroup. Then, $p_i = D_i/n$ shows an estimate to the subgroup fraction non-conforming.

The MLE of τ is the value of τ that maximises the likelihood function (Equation (2)), or equivalently, its logarithm. The logarithm of the likelihood function is

$$\log_e(L(\tau, \beta \mid D)) = k + (\log_e p_0) \sum_{i=1}^{\tau} D_i + (\log_e(1 - p_0)) \times \sum_{i=1}^{\tau} (n - D_i) + \sum_{i=\tau+1}^{\mathrm{T}} D_i \times (\log_e p_i) + \sum_{i=\tau+1}^{\mathrm{T}} (n - D_i) \times (\log_e(1 - p_i)) \tag{3}$$

where k is a predefined constant.

Since the slope of change, β, is unknown, by taking the partial derivative of Equation (3) with regard to β and equating it zero, a formula is derived for β in terms of τ that provides the maximum value for the logarithm of the likelihood function. In other words,

$$\frac{\partial}{\partial \beta} \log_e(L(\tau, \beta \mid D)) = \sum_{i=\tau+1}^{\mathrm{T}} \frac{D_i(i - \tau)}{p_0 + \beta(i - \tau)} - \sum_{i=\tau+1}^{\mathrm{T}} \frac{(n - D_i) \times (i - \tau)}{1 - p_0 - \beta(i - \tau)} \tag{4}$$

Since it is difficult to find the exact values of τ and β from Equation (4) analytically, we apply the Newton method (see Hildebrand 1987) to solve Equation (4). The optimal combination of (τ,β) obtained by the Newton method is known as the MLE of the change point. This MLE of the change point can be applied when any process fraction non-conforming control chart, including CUSUM, EWMA, and *np*, gives an out-of-control signal.

In the next section, Monte Carlo simulation is used to evaluate the accuracy and the precision performances of the proposed change-point estimator following a signal from a *np* chart.

3. Performance comparison analyses

The performances of the proposed estimator, $\hat{\tau}_{lt}$, are compared with the ones of a MLE derived for step changes proposed by Samuel and Pignatiello (2001) and the one of a MLE derived for monotonic changes suggested by Perry et al. (2007) when a linear trend disturbance is present, and the out-of-control signal comes from a Shewhart *np* control chart. The estimator proposed by Samuel and Pignatiello (2001) is derived under a step change assumption, and the estimator proposed by Perry et al. (2007) is derived under a monotonic change assumption. These are referred to $\hat{\tau}_{\mathrm{SC}}$ and $\hat{\tau}_{\mathrm{MC}}$, respectively.

3.1. False alarms

In this section, we address the handling of false alarms in the simulation model. A signal time greater than the real process change point, i.e. $T > \tau$, is referred to a genuine signal and can be used for searching the change point. Otherwise, however, if the signal time is less than the real change point, i.e. $T < \tau$, then the control chart signals before a disturbance in the process and, hence, is treated a false alarm.

In the simulation runs, the false alarm signal is not considered for the performance analysis. Whenever a signal is a false alarm, the process is assumed in control and, therefore, the control chart continues its action to monitor the process. In other words, when a false alarm happens in a simulation run at subgroup T, the control chart resumes at subgroup $T + 1$ while not altering the change-point estimation process. This is the identical approach used by Perry et al. (2007), Noorossana and Shademan (2009), and Perry and Pignatiello (2006).

3.2. Limitations of the chart parameters and the linear trend model

The linear trend model given in Equation (1) has some limitations. First, since it is proposed to model the process fraction non-conformities $0 \leq p \leq 1$, and, hence (Montgomery 1996),

$$0 \leq p_i = p_0 + \beta(i - \tau) \leq 1 \tag{5}$$

Then, for performance comparisons of the estimators in the presence of a linear trend disturbance, β values must be chosen such that Equation (5) holds. The change in the process fraction non-conformities along with its constraint is depicted in Figure 1.

Since only genuine alarms are considered, we have $T - \tau \geq 0$. Moreover, $p_0 \geq 0$ and $p_i \geq 0$; $i = 1, 2, \ldots$. Then, based on Figure 1, the value of T' can be obtained as follows:

$$p_T = p_0 + \beta\left(T' - \tau\right) = 1 \Rightarrow T' = \tau + \frac{1 - p_0}{\beta} \tag{6}$$

Now, since $T' \geq T$ is considered, β must have values such that Equation (6) on T' holds.

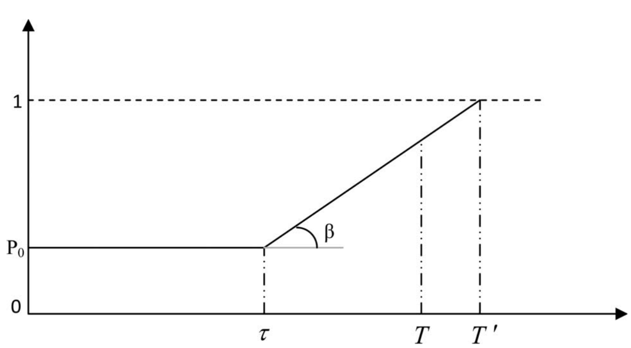

Figure 1. The constraint on the process fraction non-conforming ($0 \leq p \leq 1$).

Second, it was mentioned in the previous section that the *np* chart is used to monitor the process. Since the number of non-conforming items in each subgroup cannot be negative, i.e. $D_i \geq 0$, the lower control limit (LCL) cannot be negative either. Thus, the minimum number of each subgroup in the simulation runs is set such that Equation (7) holds.

$$\mathrm{LCL} = np_0 - 3\sqrt{np_0(1 - p_0)} \geq 0 \tag{7}$$

3.3. Performances of the MLEs for $p_0 = 0.01$ and $n = 300$

In this section, Monte Carlo simulation is used to evaluate the performances (i.e. the bias and the variability of the estimates on τ) of the change-point estimators for an in-control fraction non-conforming of $p_0 = 0.01$ with a subgroup size of $n = 300$, where the process real change point is simulated to happen at $\tau = 50$. Independent observations for simulation model are sampled from a binomial distribution with the process fraction non-conforming $p_0 = 0.01$ and subgroups of $i = 1, 2, \ldots, 50$, each having a size of $n = 300$. After subgroup 50, independent observations are simulated from a binomial distribution with the process fraction non-conforming $p_i = p_0 + \beta(i - 50)$ until the control chart signals. Based on the simulation data, the three aforementioned estimators of the process fraction non-conforming change point, i.e. $\hat{\tau}_{lt}$, $\hat{\tau}_{\mathrm{SC}}$, and $\hat{\tau}_{\mathrm{MC}}$, were then obtained. This procedure was repeated $N = 10{,}000$ times over a range of β values for each estimator. The mean squared errors, MSE, and the expected time of the change-point estimates were calculated as shown in Table 1, where the standard errors greater or equal than 0.01 are shown in parentheses. Moreover, the expected time of the first genuine alarm, $E(T)$, is the expected time at which the control chart first signals a disturbance in the process fraction non-conforming.

The results provided in Table 1 show that, except for $\beta = 0.01$, $\mathrm{MSE}(\hat{\tau}_{\mathrm{lt}})$ is smaller than both $\mathrm{MSE}(\hat{\tau}_{\mathrm{SC}})$ and $\mathrm{MSE}(\hat{\tau}_{\mathrm{MC}})$ for all other considered values of β. It means that the proposed estimator performs better than the other two estimators. Note that, in this table, as the magnitude of the slope parameter, β, increases to 0.30, the mean squared error for the three estimators decreases. However, more accurate estimates are obtained using the proposed method in almost all cases. Thus, it can be concluded from Table 1 that the proposed estimator outperforms the other two estimators and that it provides a more accurate estimate of the true process change point when a linear trend disturbance is present.

In order to evaluate and compare the precision of proposed change-point estimator with the ones of the estimators proposed by Samuel and Pignatiello (2001) and Perry et al. (2007), this procedure was repeated for a total of $N = 10{,}000$ independent simulation runs using $p_0 = 0.01$, $n = 300$, and $\tau = 50$ for each estimator. Then, the probability of the change-point estimate to lie within a certain sample from the true change point is reported in Table 2 for different values of β. In this table, the precision estimates of $\hat{\tau}_{\mathrm{MC}}$ are shown in

Table 1. Accuracy performances for three MLEs of the change point for different β values following a genuine signal from a *np* control chart when a linear trend change is present.

β	$E(T)$	$\hat{\tau}_{\mathrm{lt}}$	MSE($\hat{\tau}_{\mathrm{lt}}$)	$\hat{\tau}_{\mathrm{MC}}$	MSE($\hat{\tau}_{\mathrm{MC}}$)	$\hat{\tau}_{\mathrm{SC}}$	MSE($\hat{\tau}_{\mathrm{SC}}$)
0.01	52.189	51.167 (0.01)	1.361	38.742 (0.27)	126.804	49.413 (0.06)	0.348
0.02	51.391	50.320 (0.01)	0.102	37.967 (0.28)	144.871	49.196 (0.06)	0.648
0.03	51.090	50.007	0.004	38.335 (0.28)	136.137	49.088 (0.06)	0.834
0.04	50.984	49.835	0.027	38.882 (0.27)	123.669	49.169 (0.05)	0.693
0.05	50.950	49.665	0.112	39.039 (0.36)	120.277	49.346 (0.05)	0.429
0.07	50.949	49.669	0.109	37.742 (0.40)	150.422	49.299 (0.05)	0.493
0.08	50.947	49.755	0.050	38.007 (0.39)	143.983	49.338 (0.05)	0.440
0.09	50.950	49.809	0.036	38.582 (0.39)	130.523	49.261 (0.05)	0.548
0.11	50.943	49.888	0.012	39.040 (0.36)	120.255	49.312 (0.05)	0.476
0.13	50.949	49.933	0.004	38.306 (0.39)	136.903	49.337 (0.05)	0.441
0.15	50.949	49.947	0.003	37.712 (0.40)	151.158	49.247 (0.05)	0.569
0.19	50.947	49.910	0.007	38.487 (0.39)	132.690	49.357 (0.05)	0.415
0.23	50.946	49.565	0.189	38.453 (0.38)	133.477	49.2642 (0.05)	0.543
0.25	50.946	49.324	0.456	39.000 (0.38)	121.135	49.259 (0.05)	0.551
0.30	50.938	49.625 (0.01)	0.141	37.214 (0.41)	163.648	49.452 (0.09)	0.308

Table 2. Precision performance of the three estimators based on different values of β ($p_0 = 0.01$, $n = 300$, $\tau = 50$, and $N = 10{,}000$ independent runs).

β	0.01	0.02	0.03	0.04	0.05	0.07	0.09	0.11	0.13	0.15	0.19	0.23	0.25	0.30
$P(\lvert T - \tau\rvert = 0)$	0.35	0.48	0.62	0.71	0.79	0.89	0.93	0.94	0.92	0.95	0.95	0.95	0.95	0.95
	(0.02)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
	[0.21]	[0.57]	[0.82]	[0.78]	[0.84]	[0.86]	[0.90]	[0.91]	[0.94]	[0.95]	[0.95]	[0.87]	[0.86]	[0.51]
$P(\lvert T - \tau\rvert \le 1)$	0.45	0.90	0.92	0.92	0.93	0.94	0.94	0.94	0.94	0.94	0.94	0.95	0.95	.95
	(0.10)	(0.09)	(0.09)	(0.10)	(0.09)	(0.09)	(0.11)	(0.09)	(0.10)	(0.09)	(0.09)	(0.09)	(0.10)	(0.07)
	[0.76]	[0.99]	[1.00]	[1.00]	[1.00]	[1.00]	[0.99]	[0.99]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]
$P(\lvert T - \tau\rvert \le 2)$	0.73	0.92	0.93	0.93	0.94	0.95	0.95	0.95	0.95	0.95	0.95	0.96	0.96	0.96
	(0.21)	(0.20)	(0.20)	(0.23)	(0.20)	(0.22)	(0.22)	(0.19)	(0.21)	(0.21)	(0.21)	(0.20)	(0.21)	(0.17)
	[0.96]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]
$P(\lvert T - \tau\rvert \le 3)$	0.79	0.94	0.94	0.95	0.96	0.96	0.96	0.96	0.96	0.96	0.96	0.97	0.97	0.97
	(0.30)	(0.28)	(0.29)	(0.31)	(0.29)	(0.29)	(0.30)	(0.29)	(0.29)	(0.29)	(0.29)	(0.28)	(0.29)	(0.25)
	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	1.00]	[1.00]	[1.00]	[1.00]
$P(\lvert T - \tau\rvert \le 5)$	0.86	0.95	0.95	0.96	0.96	0.97	0.97	0.97	0.97	0.97	0.97	0.98	0.98	0.98
	(0.44)	(0.38)	(0.41)	(0.44)	(0.41)	(0.40)	(0.43)	(0.42)	(0.43)	(0.21)	(0.42)	(0.41)	(0.43)	(0.38)
	[1.00]	[1.00]	[1.00]	[1.00	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]
$P(\lvert T - \tau\rvert \le 7)$	0.91	0.95	0.96	0.96	0.96	0.97	0.97	0.97	0.98	0.98	0.98	0.98	0.98	0.98
	(0.54)	(0.51)	(0.54)	(0.56)	(0.54)	(0.51)	(0.55)	(0.55)	(0.54)	(0.31)	(0.54)	(0.54	(0.55)	(0.49)
	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]
$P(\lvert T - \tau\rvert \le 9)$	0.94	0.95	0.96	0.96	0.96	0.97	0.97	0.97	0.98	0.98	0.98	0.98	0.98	0.98
	(0.64)	(0.61)	(0.69)	(0.65)	(0.64)	(0.1)	(0.64)	(0.65)	(0.62)	(0.59)	(0.63)	(0.63)	(0.64)	(0.60)
	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]
$P(\lvert T - \tau\rvert \le 13)$	0.95	0.95	0.96	0.96	0.96	0.97	0.97	0.98	0.98	0.98	0.98	0.99	0.98	0.8
	(0.73)	(0.71)	(0.73)	(0.73)	(0.75)	(0.72)	(0.73)	(0.75)	(0.72)	(0.60)	(0.74)	(0.72)	(0.74)	(0.70)
	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]
$P(\lvert T - \tau\rvert \le 17)$	0.96	0.95	0.96	0.96	0.96	0.97	0.98	0.98	0.98	0.98	0.98	0.99	0.99	0.99
	(0.78)	(0.78)	(0.78)	(0.79)	(0.80)	(0.77)	(0.79)	(0.80)	(0.77)	(0.76)	(0.79)	(0.78)	(0.80)	(0.75)
	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]
$P(\lvert T - \tau\rvert \le 19)$	0.96	0.95	0.96	0.96	0.96	0.97	0.98	0.98	0.98	0.98	0.98	0.99	0.99	0.99
	(0.82)	(0.80)	(0.81)	(0.81)	(0.83)	(0.78)	(0.81)	(0.83)	(0.81)	(0.78)	(0.81)	(0.80)	(0.83)	(0.77)
	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[.00]	[1.00]	[1.00]	[1.00]
$P(\lvert T - \tau\rvert \le 20)$	0.96	0.95	0.96	0.96	0.96	0.97	0.98	0.98	0.98	0.98	0.98	0.99	0.99	0.99
	(0.83)	(0.81)	(0.81)	(0.82)	(0.84)	(0.79)	(0.81)	(0.84)	(0.82)	(0.79)	(0.81)	(0.81)	(0.84)	(0.79)
	[1.00]	[1.00]	[1.00]	[1.00]	1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]

parentheses and the precision estimates of $\hat{\tau}_{lt}$ are shown in brackets.

Based on the results in Table 2, the $\hat{\tau}_{lt}$ estimator provides a more or at least equal precise estimate in comparison with $\hat{\tau}_{\mathrm{SC}}$ and $\hat{\tau}_{\mathrm{MC}}$ if the changes follow a linear trend model. Moreover, regarding to the results obtained in Table 2, the precision estimates of the estimators are plotted in Figures 2–7, where they show precision estimate values versus possible slope of change trends for specified tolerances. Further, for each value of β, three values of different estimators are compared. In these figures, the precision performances of the proposed estimator are shown in comparison with the other estimators in the presence of linear disturbance.

Figure 2 shows that, for process change with $\beta = 0.03$, the estimated probability of correctly identifying the time of the process change using the proposed estimator is 82%, whereas the estimated probabilities of $\hat{\tau}_{\mathrm{SC}}$ and $\hat{\tau}_{\mathrm{MC}}$ are 62% and 1%, respectively. It should be noted that the precision provided by the proposed estimator of the accurate change point (limiting value of the probability $P(|T - \tau| = 0)$ was not absolutely better than $\hat{\tau}_{\mathrm{SC}}$, where, for some values of β, the precision performance of $\hat{\tau}_{\mathrm{SC}}$ is equal or even better than the ones of the proposed estimator. For example, the estimated limiting values for this

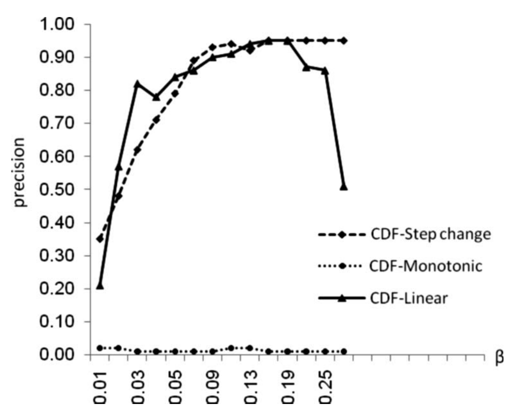

Figure 2. Precision of estimators for the estimated accurate change point $P(|T - \tau| = 0)$.

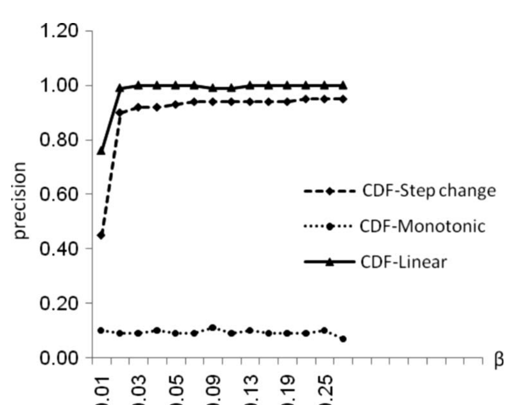

Figure 3. Precision of estimators for tolerance 1 subgroup $P(|T - \tau| \leq 1)$.

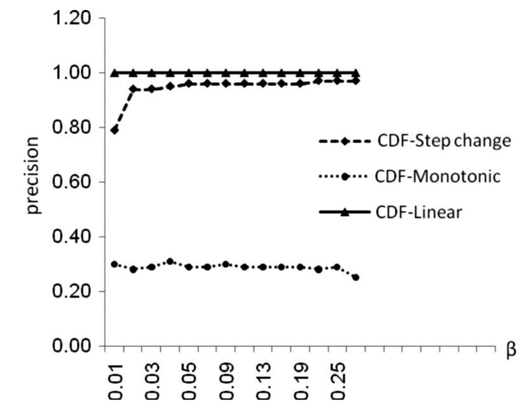

Figure 4. Precision of estimators for tolerance 3 subgroups $P(|T - \tau| \leq 3)$.

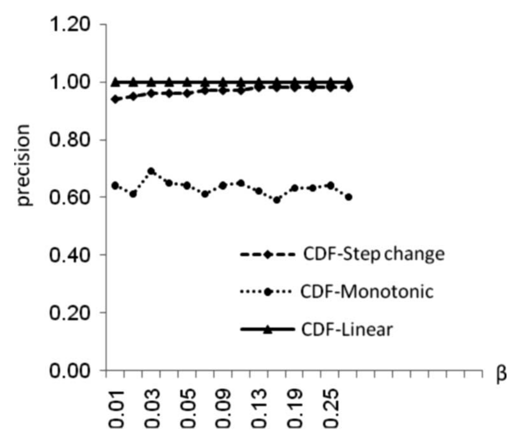

Figure 5. Precision of estimators for tolerance 9 subgroups $P(|T - \tau| \leq 9)$.

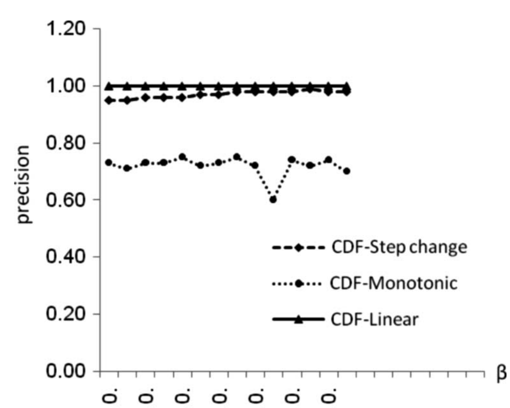

Figure 6. Precision of estimators for tolerance 13 subgroups $P(|T - \tau| \leq 13)$.

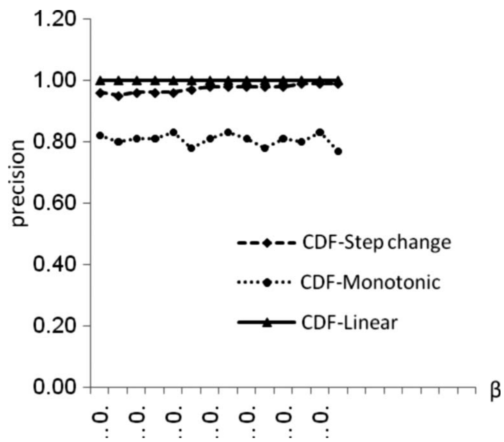

Figure 7. Precision of estimators for tolerance 17 subgroups $P(|T - \tau| \leq 17)$.

probability with change slope of $\beta = 0.11$, using $\hat{\tau}_{SC}$ and $\hat{\tau}_{lt}$, are 0.82 and 0.49, respectively. However, for other limiting values of the probability, $\hat{\tau}_{SC}$ has perfect precision.

The precision provided by the proposed estimator for the true change point within 1 subgroup, which is in the presence of linear disturbance, is better than the other two estimators. While increasing the magnitude of the change slope in the process fraction non-conforming, it is observed that the precision preference of the proposed estimator is as good as that of the accurate performance as shown in Table 1. Moreover, the precision of $\hat{\tau}_{SC}$ is improved by the increase in β values. As observed in Figures 2–7, by increasing the tolerance value of the precision, $\hat{\tau}_{MC}$ has a rapid trend, never having an acceptable precision in comparison with the other two estimators. Moreover, it can be seen that the estimation of the change point using the proposed estimator is within 3 subgroups of the true process change point in all the simulation runs, whereas the other two estimators do not have such precision within 20 subgroups of the true process change point.

3.3.1. *The parameter constraint*

In this section, the parameter constraints (Equations (5) and (7)) that was mentioned in Section 3.2 are considered, where the process change point is simulated to happen at $\tau = 50$. Independent observations are simulated from a binomial distribution the with process fraction non-conforming of $p_0 = 0.01$ and subgroup size of $n = 300$. Using Equation (6), the value of T' is obtained as

$$T' = 50 + \frac{1 - 0.01}{\beta} \tag{8}$$

Table 3. Limitation of the β value.

β	$E(T)$	$\hat{\tau}_{lt}$	Standard error	$MSE(\hat{\tau}_{lt})$	Count
1.50	46.866	39.00	0.00	1.00	8661
2.00	46.864	39.00	0.00	1.00	8641
2.50	46.866	39.00	0.00	1.00	8661

Table 3 shows the simulation results for ineligible β values that have no suitable circumstances for comparison between estimators.

A *count* variable enumerates the simulation runs that the control chart signal time, T, is greater than the calculated T'. If the *count* value of the simulation runs is considerable (for example, greater than or equal to 50) for each β value, then this β value is not suitable for the comparison between the estimators. The above results are repeatable for other estimators as well. With control charts that are more precise relative to the *np* chart, like CUSUM and EWMA, the accuracy, and the precision performance analysis of the estimating change-point methods can be improved.

The second constraint is concerned on the size of the subgroups such that the LCL becomes non-negative. Regarding this constraint, for the specific simulation runs at hand, the minimum number of a subgroup can be determined using Equation (7) as

$$\begin{aligned} \mathrm{LCL} &= n \times 0.01 - 3\sqrt{n \times (0.01) \times (1 - 0.01)} \geq 0 \\ &\Rightarrow n \geq 300 \end{aligned} \tag{9}$$

In other words, the minimum subgroup size required for the *np* chart to have non-negative LCL is 300. Samuel and Pignatiello (2001) have also indicated that as the size of the subgroup increases, the accuracy and the precision performance of the estimator improve. It is up to the process engineering to determine the subgroup size considering economic limitation, lower limitation, and the minimum precision.

It should be noted that although the batches are mostly made small today, there are still many manufacturing processes that make large batches. Moreover, in the proposed methodology, one can make the batches smaller and obtain negative LCL that can be assumed zero.

4. Conclusion

When a control chart signals an out-of-control condition, a search begins to identify and, hence, to eliminate the source(s) of the special cause. The time when a special cause manifests itself into a process is referred to a change point. Estimation of the genuine time and the real source of the disturbance cause(s) in the process fraction non-conforming is valuable for process engineers and technicians who would like to gain more ease and quick identification of the variables and/or procedures that might cause a change in their processes. In this article, an estimator based on the maximum likelihood was proposed that helps to identify the change point when a disturbance of linear nature shifts the process fraction non-conforming. Manufacturing processes in which special causes can happen due to the linear trend disturbances in fraction non-conforming involve gradual tool wear, machine depreciation, workers' fatigue, filters that become dirty over time, or any other time-related factors that can affect the quality of produced items. Moreover, the performance of the proposed method was compared with the ones of two other available estimators that were developed by Samuel and Pignatiello (2001) and Perry et al. (2007) in the presence of step change and monotonic change type, respectively. The results of this research showed that the MLE obtained for the linear trend change has better performance than the ones for the step change and monotonic change type when a linear trend disturbance is present. We note that if a linear process change is simulated, the model that proposes a linear process change will give the best results. In practice, at the appearance of the out-of-control signal, one does not know the mathematical function of process disturbance. Usually, when the out-of-control signal is detected, the diagnostic is started. One of the diagnoses can suppose a linear process change. In this case, the proposed method is useful in SPC.

The following ideas may be considered for future research:

(1) Using more accurate methods, like CUSUM and EWMA, to monitor the process fraction non-conforming may result in better estimates of the process change point.
(2) In addition to the 'above upper control limit' rule that causes an out-of-control signal, Western Electric or other sensitising rules may also be employed to improve the precision of the future change-point estimator.
(3) The proposed estimator has been compared with the step change and monotonic change type estimators when a linear trend disturbance is present. In this case, we showed the proposed estimator outperforms the other two estimators available in the literature. The comparison study for other kinds of practical changes, for example, step or periodic disturbance, may be investigated in future.
(4) While the proposed methodology has been tested using simulation, finding a real manufacturing case study may be considered in the future.

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