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On Fixed Points of Digraphs Over Lambert Type Map

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Abstract Define $f(y) = y^2h^y$, where $h \in (Z/mZ)$, the discrete Lambert Type Map(DLTM). For a set of pairs of vertices and edges DLTM diagraphs are obtained in which the vertices are allocated are from a whole range of residues modulo a fixed integer and edges are built when $f(y) = v(mods^k)$ is solvable in t and in terms of diophantine equation as well. In this paper we proposed new results for digraphs over Lambert type map for fixed point and for multiple order.

Keywords Fixed Point, Discrete Lambert Type Map, Residues

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1 Introduction

It is well-known that the solution of exponential type function under congruence is not always easy. In fact, it is always a challenging problem in number theory. To find the solution of equations in which unknown appearing in exponential terms, the Lambert function $W(z)e^{W(z)}$ is helpful and have been used frequently by many researchers. These functions already have been studied in [4,5] and in [17] as well. This is, sometime defined as well, $z = W(z)e^{W(z)}$, where z is a complex number. Before finding the fixed point, one should understand the meanings of fixed point with respect to the environment, we are using for. For a detail understanding of fixed points and integer classes, we suggest to read the references [1-3], [4,5,17],[18-20] and then [6-16], so that the reader could enjoy the reading of our proposed results.A graph is an ordered pair G(V, E) which consists on two sets V and E, where V is set of points named as set of vertices taking as the residues of any given fixed integer and E is a set of edges, obtained by using DLTM and a graph in which edges have direction is called directed graphs or digraphs

Definition 1. *Graph A graph is an ordered pair $G(V, E)$ which consists on two sets V and E, where V is set of points named as set of vertices and E is a set of edges.*

Definition 2. *Digraph A graph in which edges have direction is called directed graphs or digraphs.*

2 Fixed Point.

: A number β is said to be fixed point of DLTM iff $\beta^2 g^\beta \equiv \beta(mod m)$ for a positive integer m. The following results are elaborating fixed points and image structures for specific numbers.

Theorem 1. *If $h \equiv 1 \bmod s$ and $h + x = s^2 + 2$ then x is fixed point of the graph (h, s^2). In this case, 0 and x are the only two fixed points.*

Proof.

$$\text{Let } h + x = s^2 + 2 \quad (1)$$
$$\text{then } x = s^2 + 2 - h \quad (2)$$

As $h \equiv 1 \bmod s$, so there must exist an integer t such that $h = 1 + ts$. Putting in equation (2.1.2),

$$\begin{aligned} x &= s^2 + 2 - (1 + ts) \\ &= s^2 + 2 - 1 - ts \\ &\equiv 1 - ts \,(\text{mod } s^2) \\ \text{or } x &\equiv 1 - ts \,(mod\ s^2) \quad (3) \end{aligned}$$

Now using equation (2.1.3) in $f(x) = x^2 h^x \bmod s^2$, we get

$$\begin{aligned} f(x) &\equiv ((1 - ts)^2 (1 + ts)^{(1-ts)}) \bmod s^2 \\ &\equiv ((1 + t^2 s^2 - 2ts)(1 + ts(1 - ts) + terms\ \ involving\ \ s^2)) \bmod s^2 \\ &\equiv ((1 + t^2 s^2 - 2ts)(1 + ts - t^2 s^2 + terms\ \ involving\ \ s^2)) \bmod s^2 \\ &\equiv ((1 - 2ts)(1 + ts)) \bmod s^2 \\ &\equiv (1 - 2ts + ts - 2t^2 s^2) \bmod s^2 \\ &\equiv (1 - ts) \bmod s^2 \\ &\equiv x \bmod s^2 \end{aligned}$$

This completes the proof. □

The Figure 2.1 depicts the above result.

Theorem 2. *Let f be a discrete Lambert Type Map. For an odd prime s, if $h = s - 2$, then h is always a fixed point of f.*

Proof. Let $f(t) = t^2 h^t (mod\ s)$ and $h = s - 2$.

$$\begin{aligned} f(s-2) &\equiv ((s-2)^2 (s-2)^{s-2})\ (mod\ s) \\ &\equiv (s-2)^{2+s-2}) mod\ s) \\ &\equiv (s-2)^s (mod\ s). \quad (4) \end{aligned}$$

Now by Euler's Theorem, we know that $a^s \equiv a(mod\ s)$. But then the equation (2.1.4) yields,

$$f(s-2) \equiv (s-2)(mod\ s). \quad (5)$$

Hence $s - 2$ is a fixed point. □

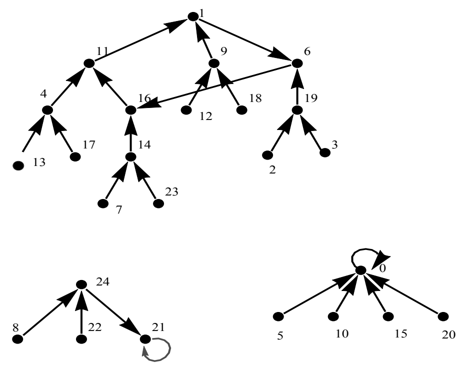

Figure 1. Shows the diagraph $G(6, 5^2)$

3 Multiplicative Order

Let g be any fixed integer and integer m > 0*. Recall that a multiplicative order modulo m is a least positive integer* β *such that* $g^{\beta} \equiv 1 \mod m$*. This is denoted by* $Ord_m g = \beta$*. By incorporating this definition, some other foxed points of the DLTM can be calculated. These are given in the following theorems as under:*

Theorem 3. *Let f be a DLTM. If* $Ord_{s^k} h = s^{k-1},\ k > 1$*, then the diagraph* $G(h, s^k)$ *have two fixed points namely* 0 *and* $s^2 - s + 1$*. Also, all multiples of s maps on zero and make an independent component.*

Proof. Since $Ord_{s^k} h = s^{k-1},\ k > 1$, so in particular $Ord_{s+1^2} h = s+1$. Consider

$$f(s^2 - s + 1) = ((s^2 - s + 1)^2 (s+1)^{(s^2-s+1)}) \mod s^2 \tag{6}$$

As $(s+1, s^2) = 1$, so by Eular's Theorem, we have $(s+1)^{\phi(s^2)} \equiv 1 \mod s^2$. Putting in equation (2.2.1). we get

$$\begin{aligned} f(s^2 - s + 1) &\equiv ((s^2 - s + 1)^2 (s+1)) \mod s^2 \\ &\equiv (1-s)^2(1+s) \mod s^2 \\ &\equiv ((1 - 2s + s^2)(s+1)) \mod s^2 \\ &\equiv (1-2s)(1+s) \mod s^2 \\ &\equiv s + 1 - 2s^2 - 2s \mod s^2 \\ &\equiv 1 - 2s^2 - s \mod s^2 \\ &\equiv (1-s) \mod s^2 \\ &\equiv (1 - s + s^2) \mod s^2 \end{aligned}$$

so $(s^2 - s + 1)$ is fixed point. Also, it can easily be seen that $f(s^2\beta) \equiv (s^2\beta)^2 g^{s^2\beta} \equiv 0 \mod s^2$ □

Figure 2.2 depicts the digraph $G(12, 11^2)$*,where* $|12| = 11$ *and all multiple of 11<121 maps on zero. It has 4 cycles and the numbers 0 and 111 are the only fixed points.*

Proposition 1. *Let f be a DLTM and s be any odd prime.* $Ord_{s^k} h = 2$ *if and only if* $h = s^k - 1$ *for any integer k. In this case, 0 is only fixed point of the map.*

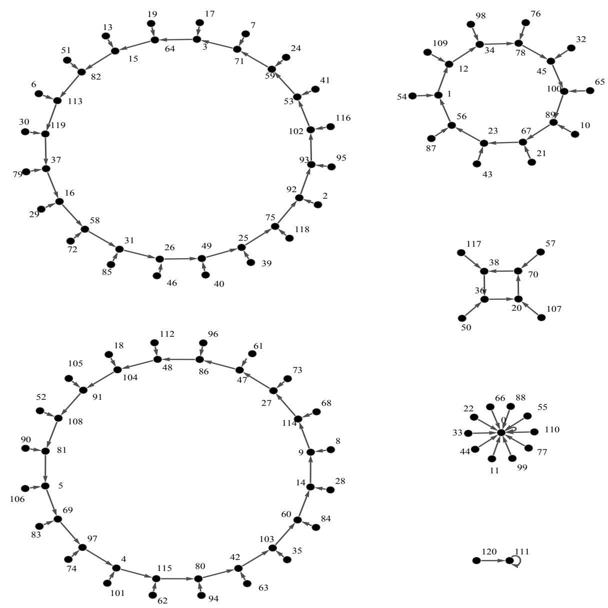

Figure 2. Shows the diagraph $G(12, 11^2)$

Proof. Let $Ord_{s^k} h = 2$. Then 2 is the least positive integer such that $h^2 = 1 \ (mod\ s^k)$. This means that $h^2 = 1 \ (mod\ s^k)$ or

$$\begin{aligned} h^2 &\equiv (-1)^2 (mod\ s^k) \\ &\equiv s^k - 1)^2 \ (mod\ s^k) \\ \text{or } h^2 &\equiv (s^k - 1)^2 \ (mod\ s^k) \end{aligned}$$

This surely gives $h = s^k - 1$.
Conversely, if we assume $h = s^k - 1$, then

$$\begin{aligned} h^2 &\equiv (s^k - 1)^2 \ (mod\ s^k) \\ &\equiv (-1)^2 \ (mod\ s^k) \\ &\equiv 1 \ (mod\ s^k) \end{aligned}$$

Also, note that for any vertex h, $h^2 g^h \equiv g^h \not\equiv h \ (mod\ s^k)$. Thus, 0 is the only fixed point. □

If we take $s = 3, k = 3, h = 26$, then order of 26 is 2 mod 27, so Figure 2.3 elaborate Proposition 1.

Proposition 2. *If t is a fixed point of n, then there must be some y in $G(n)$ such that*

$$y \equiv h^t \ (mod\ n) \ \ and \ \ ty \equiv 1 \ (mod\ n).$$

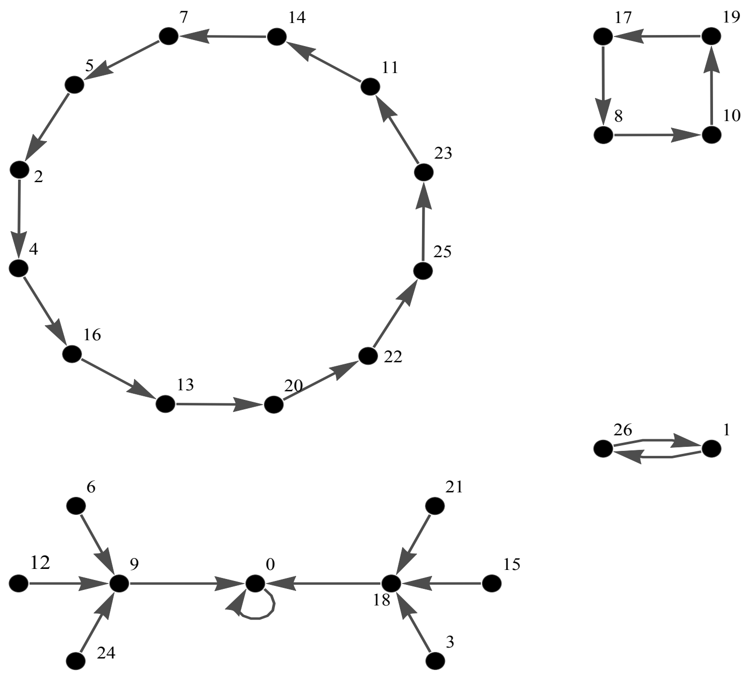

Figure 3. Shows the diagraph $G(26, 27)$

Proof. Let t be a fixed point of $G(n)$

$$
\begin{aligned}
f(t) &= t^2h^t \\
t &\equiv t^2h^t \ (mod\ n) \\
t^2h^t - t &\equiv 0\ (mod\ n) \\
t(tg^t - 1) &\equiv 0\ (mod\ n) \\
t &\equiv 0\ (mod\ n) \\
\text{or} \quad tg^t - 1 &\equiv 0\ (mod\ n). \\
\text{or} \quad tg^t &\equiv 1\ (mod\ n).
\end{aligned}
$$

This clearly shows that t and g^t are the multiplicative inverse modulo n. Thus there must exist some integer y such that $y \equiv h^t\ (mod\ n)$ and this implies that $ty \equiv 1\ (mod\ n)$. This means that if t is a non-zero fix point then t^{-1} is also a fix point of n. Moreover, all units of m are the fixed of m. □

Corollary 1. *Let α be a fixed point of n, then either $\alpha \equiv 0$ or α is a unit of n.*

Proof. We know that, $\alpha, \beta \in RRS\ \ of\ \ s$ are units if and only if

$$\alpha\beta \equiv 1\ (mod\ s),$$

$f(t) = t^2h^t$, when α is fixed point

$$
\begin{aligned}
\alpha^2h^\alpha &= \alpha\ (mod\ s) \\
\alpha(\alpha h^\alpha - 1) &= 0\ (mod\ s),
\end{aligned}
$$

$\alpha \equiv 0 \pmod{s}$ or $\alpha h^{\alpha} \equiv 1 \pmod{s}$.
Now if $\alpha h^{\alpha} \equiv 1 \pmod{s}$, we take $h^{\alpha} = \beta$ for sake of convenience.
There $\alpha h^{\alpha} \equiv 1 \pmod{s}$ yields the $\alpha\beta \equiv 1 \pmod{s}$ or α is a unit of s. □

Corollary 2. *An integer t is a fixed of G(n)* $\Leftrightarrow$ $th^{t} \equiv 1 \pmod{n}$

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