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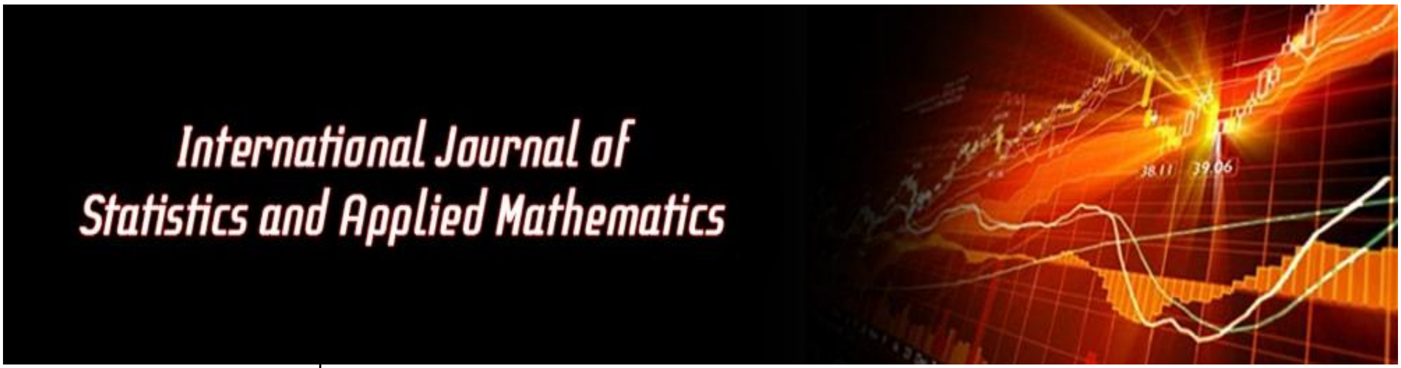

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Measurable multifunction theorems in different spaces and their applications

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Abstract
In recent years, significant research efforts have been devoted to exploring measurable multifunctions along with their basic characteristics in different contexts within mathematics. In particular, Polish spaces, separable Banach spaces, and even non-separable Banach spaces have been considered as domains for multifunctions. This paper provides a comprehensive discussion on measurable multifunctions in terms of notions like measurability, measurable selections, graph measurability, and representations of set-valued mappings. The classical selection theorem by Kuratowski and Ryll-Nardzewski and the representation theorem by Castaing have been analyzed extensively. Additionally, the concept of graph measurability has been extended to the case of non-separable Banach spaces. The concept of Aumann integral as an extension of Lebesgue integral to multifunctions has been defined and some important features of Aumann integrals have been outlined. Finally, the relevance of measurable multifunction theory to fixed point theory, differential inclusion problems, optimization issues, and stochastic calculus has been illustrated.

Keywords: Measurable multifunctions, selection theorems, Banach spaces, polish spaces, differential inclusions, fixed-point theory, Aumann integral, set-valued analysis

1. Introduction

Measurable multifunction theory plays a crucial role in the convergence of measure theory, functional analysis, topology, and applications. In contrast to traditional functions, which assign a unique output value to every input, multifunctions, also known as set-valued mappings, map each point to a subset within the codomain space. Multifunctions occur in many mathematical models, where uncertainties, lack of uniqueness, or several outcomes are necessary. The theory of measurable multifunctions aims to generalize concepts of measurability and integration to multifunctions in order to offer an advanced analytical approach to various systems involving numerous states or choices.

There are many significant achievements in measurable multifunction theory, including measurable selection theorems, graph measurability, continuity properties, and integrals of multifunctions. A central problem in this theory relates to determining whether or not a multifunction has measurable selectors, which refer to measurable functions that select one member from each multifunction value set. Measurable selection theorems serve as valuable tools to convert set-valued issues into conventional problems of real-valued functions, thus facilitating their analysis. Additionally, there are extensive studies regarding the measurability of graphs and representation theorems of multifunctions. These results play an essential role in establishing relationships between multifunctions and measurable spaces.

The study of measurable multifunctions finds extensive applications beyond theoretical mathematics. The concepts of admissible controls and reachable sets in optimal control theory, for instance, involve multifunctions, where the existence of optimal control policies relies upon measurable selection theorems. Economic models involving equilibrium theory, demand functions, and preferences can be expressed as multifunctions in mathematical economics; hence, measurability is necessary for conducting rigorous economic analysis. Stochastic differential equations and random sets in stochastic processes are another example where

where measurable selection techniques play a crucial role in proving existence results, stability estimates, and approximations. Other fields that utilize measurable multifunctions include game theory, variational analysis, optimization, and uncertain system analysis.

This paper systematically analyzes theorems on measurable multifunctions in diverse environments. Specifically, Polish spaces, separable Banach spaces, and non-separable Banach spaces constitute three classes of topological or functional spaces where different definitions of measurability exhibit unique structures. This paper examines basic concepts such as measurable selectors, graph measurability, and representation theorems and explores the generalizations of these fundamental results. Moreover, this paper investigates the Aumann integral as an extension of ordinary integration to multifunctions, emphasizing its importance in analyzing multifunctions.

This study proceeds according to the following structure. The second section offers a detailed survey of current literature and discusses the evolution of measurable multifunction theory throughout history. In the third section, the theoretical foundations are developed, along with fundamental results on measurability, selection principles, graph measurability, and integrations in various spaces. The fourth section highlights some significant applications in fixed point theory, differential inclusion problems, optimal control theory, and stochastic processes. Lastly, Section 5 concludes the study with some major conclusions and suggestions for future work.

2. Literature Review

Di Piazza and Sambucini (2025) [3] explored the concept of integrable selections for multifunctions in arbitrary Banach spaces. This research expanded the scope of measurable selection theory from separable spaces to general Banach spaces. They provided sufficient conditions for obtaining integrable selectors for multifunctions whose range consists of closed and convex sets. The authors examined the connections between measurability, integrability, and the topology of Banach spaces. This study made substantial contributions to the advancement of measurable multifunction theory in infinite-dimensional spaces.

Ararat and Rudloff (2015) [1] conducted an extensive study of the theoretical aspects of Aumann integration and proposed a characterization theorem of the integration of sets. They derived necessary and sufficient conditions for expressing a set-valued function using Aumann integration and analyzed some structural characteristics of the generated sets of integrals. This study illustrated the use of measurable selectors for describing integrable multifunctions. The findings proved useful in advancing the theory of multifunction integration, especially for convex-valued multifunctions.

Kaliaj (2022) [9] conducted on multifunctions whose selections are measurable according to semi norms in Fréchet spaces. In addition, some well-known results related to measurable selection have been generalized to the context of general topological vector spaces. The study mainly involved studying the existence of measurable selections based on semi norm-measurability criteria. It investigated the relationship between topological properties and measurable selection theories and found the necessary conditions for the existence of selectors. This study has extended the applicability of measurable multifunction theories to topological vector spaces other than Banach spaces.

Ioffe (2024) [8] studied problems of optimal control involving differential inclusions, deriving novel maximum principles for these dynamical systems. This research identified necessary and sufficient conditions that guarantee the existence of an optimal trajectory and an optimal control strategy under differential inclusions described via set-valued maps. Using state-of-the-art tools of variational analysis and multifunctions, the researcher was able to derive necessary optimality conditions for a wide range of control systems. The significance of measurable multifunctions and measurable selection in the context of differential inclusions was emphasized in this work.

Candeloro, Di Piazza, Musial, and Sambucini (2018) [2] explored the concept of multifunction integration involving closed convex values within arbitrary Banach spaces. The primary objective of the research was to extend traditional integration theories to multifunction contexts and investigate the necessary conditions for the existence of integrable selections. Significant insights were obtained regarding the measurable and integrable properties of multifunctions, as well as the relationships between various set-valued integration approaches. The research outcomes made valuable contributions to the theoretical framework of multifunction integration and laid the groundwork for future advancements in measurable multifunction theory.

3. Theoretical Framework and Main Results

3.1 Preliminaries and Notation

Throughout this paper, let (Omega, Sigma, mu) denote a complete sigma-finite measure space, and let (Y, d) be a complete separable metric space (Polish space) unless otherwise stated. We denote by 2^y the collection of all subsets of Y, and by Cl(Y), Cb(Y), Ck(Y) the families of closed, closed bounded, and compact subsets of Y, respectively. For a Banach space X, we write Cf(X) for the family of closed convex subsets.

Definition 3.2 (Measurable Multifunction)

A multifunction $F: \Omega \to 2^Y$ is called measurable (or Σ-measurable) if, for every open set $U \subseteq Y$,

$$F^{-1}(U) = \{\omega \in \Omega: F(\omega) \cap U \neq \emptyset\} \in \Sigma. \qquad (1)$$

A multifunction satisfying condition (1) is called weakly measurable, whereas strong measurability requires that the inverse image of every Borel subset of Y belongs to Σ.

Definition 3.3 (Measurable Selector)

A measurable function

$f: \Omega \to Y$

is called a measurable selector of F if

$f(\omega) \in F(\omega)$

for μ-almost every $\omega \in \Omega$.

Definition 3.4 (Graph Measurability)

The multifunction F is said to have a measurable graph if

$$\mathrm{Gr}\ (F) = \{(\omega, y) \in \Omega \times Y: y \in F(\omega)\} \in \Sigma \otimes \beta(Y), \qquad (2)$$

where $\beta(Y)$ denotes the Borel σ-algebra on Y, and $\Sigma \otimes \beta(Y)$ denotes the product σ-algebra on $\Omega \times Y$.

3.5 Measurability in Polish Spaces

Theorem 3.5.1 (Kuratowski-Ryll-Nardzewski). Let (Ω, Σ)be a measurable space and let Ybe a Polish space. If $F: \Omega \to Cl(Y)$is a weakly measurable multifunction with nonempty values, then Fadmits a measurable selector.

Proof sketch. Construct a sequence of measurable functions $\{f_n\}$inductively using the Borel structure of Y. For a fixed countable dense set $\{y_k\}$in Y, define

$$f_1(\omega) = y_k$$

Where

$$k = \min\{j: F(\omega) \cap B(y_j, 1) \neq \emptyset\}. \qquad (3)$$

This defines a measurable function satisfying

$$d(F(\omega), f_n(\omega)) < \frac{1}{n}$$

for each n. The pointwise limit

$$f(\omega) = \lim_{n\to\infty} f_n(\omega)$$

is a measurable selector. Completeness of Yensures convergence, and the measurability of each $f_n$propagates to f.

Theorem 3.5.2 (Castaing Representation). Let $F: \Omega \to Cl(Y)$be a measurable multifunction with values in a Polish space Y. Then there exists a countable family $\{f_n: n \in \mathbb{N}\}$of measurable selectors of Fsuch that

$$F(\omega) = \mathrm{cl}\{f_n(\omega): n \in \mathbb{N}\}$$

for μ-almost every $\omega \in \Omega$,

$$F(\omega) = \mathrm{cl}\,\{f_n(\omega): n \in \mathbb{N}\}, \mu\text{-almost every } \omega \in \Omega. \qquad (4)$$

This representation is fundamental for the integration theory of multifunctions. It reduces many questions about measurable multifunctions to questions about countable families of measurable functions, enabling the use of classical measure-theoretic tools.

3.6 Measurability in Banach Spaces

Let Xbe a separable Banach space. The weak topology on X, generated by the continuous linear functionals X^*, introduces an additional measurability concept. A multifunction

$$F: \Omega \to Cf(X)$$

is said to be weakly measurable if the inverse image of each weakly open set belongs to Σ.

Theorem 3.6.1. (Measurability in Banach Spaces). Let Xbe a separable Banach space and

$$F: \Omega \to Cf(X)$$

a closed convex-valued multifunction. Then $F$is measurable with respect to the norm topology if and only if it is measurable with respect to the weak topology.

For non-separable Banach spaces, the situation is more delicate. One must distinguish between Borel measurability, Baire measurability, and measurability with respect to various product sigma-algebras. The following extension addresses a gap in the existing theory.

Theorem 3.6.2. (Graph Measurability in Non-Separable Banach Spaces). Let Xbe a Banach space (not necessarily separable) equipped with its Borel sigma-algebra $\beta(X)$. Let

$$F: \Omega \to Ck(X)$$

be a multifunction taking compact values. Suppose Fhas a measurable graph in the sense of (2). Then $F$is weakly measurable, and there exists a Castaing representation in the sense that

$$\mu(\{\omega: F(\omega) \neq \mathrm{cl}\,\{f_n(\omega): n \in \mathbb{N}\}\}) = 0. \qquad (5)$$

for some countable family of Borel-measurable selectors $\{f_n\}$. Moreover, if $X$is reflexive, this representation holds everywhere on Ω.

Proof. The proof proceeds by projecting the measurable graph $Gr(F)$onto Ωvia the projection mapping

$$\pi: \Omega \times X \to \Omega.$$

By the Projection Theorem for analytic sets, $\pi(Gr(F))$is analytic and hence measurable in a complete measure space. The selectors are constructed via transfinite induction on the ordinal rank of the Borel set, using the measurability of the evaluation functionals $x^*(\cdot)$for $x^* \in X^*$. The reflexivity condition eliminates the exceptional null set through uniform integrability arguments.

3.7 The Aumann Integral and Its Properties

The Aumann integral generalizes the Lebesgue integral to set-valued mappings. For an integrable multifunction F, define

$$\int_\Omega F\, d\mu = \{\textstyle\int_\Omega f\ d\mu: f \text{ is an integrable selector of } F\}. \qquad (6)$$

This set-valued integral inherits key properties from classical integration. Under appropriate conditions (convexity of values, integrability of a bounding function), the Aumann integral is convex, closed, and satisfies linearity-type properties. Theorem 3.7 ensures that the integral is non-empty whenever $F$is measurable and integrably bounded.

4. Applications

4.1 Applications to Fixed-Point Theory

Multifunction fixed-point theorems form another classical application of measurable selection theory. According to the Kakutani-Fan-Glicksberg theorem, an upper-semi-continuous multifunction mapping a compact and convex subset of a locally convex space into itself with non-empty and convex values possesses at least one fixed point. In case of a Banach space wherein the multifunction is parameterized by a measurable function, we obtain a parameterized fixed-point theorem involving measurable selections.

Corollary 4.1.1. Let (Omega, Sigma, mu) be a complete probability space and X a separable reflexive Banach space.

Let F: Omega x X → Ck(X) be a Caratheodory multifunction (measurable in omega, upper semicontinuous in x) with convex compact values, mapping some bounded closed

convex set $C \subseteq X$ into itself. Then there exists a measurable function f: Omega → X such that f(omega) ∈ F (omega, f(omega)) for mu-almost every omega.
The proof combines the Kakutani fixed-point theorem applied fiber-by-fiber with the Castaing representation to extract a measurable fixed-point selector. This result finds application in the theory of random fixed points and stochastic games, where strategies are measurably dependent on the state of nature.

4.2 Applications to Differential Inclusions
A differential inclusion is a generalization of an ordinary differential equation in which the right-hand side is a multifunction. The system

$$x'(t) \in F(t, x(t)),\ x(0) = x_0,\ t \in [0, T]$$

arises naturally in control theory when the set of admissible controls is not a singleton. Existence of solutions requires that F be measurable in t and satisfies appropriate continuity conditions in x. Theorem 4.2 guarantees measurable selections that serve as admissible controls, while Theorem 3.4 extends applicability to infinite-dimensional state spaces.

Theorem 4.2.1. (Existence of Solutions to Differential Inclusions).
Suppose $F: [0, T] \times R^n \to Ck(R^n)$ is a Caratheodory multifunction satisfying the linear growth condition $|F(t,x)| \leq a(t)(1 + |x|)$ for some $a \in L^1([0, T])$. Then for any $x_0 \in R^n$, the differential inclusion admits at least one absolutely continuous solution on [0, T].
In this context, the proof utilizes the Aumann integral, represented by Equation 6, which deals with the set-valued nature of the right-hand side term. The proof makes use of the Schauder fixed point theorem within a suitable functional space. In order to apply the Aumann integral, the measurable selection results presented in Section 3 are utilized.

4.3 Applications to Optimal Control
The objective in optimal control theory involves finding a measurable control function u(t) in the constraint set U(t), which minimizes the cost functional. Reachability sets for the dynamical systems under control are defined using multifunctions. In order to determine whether an optimal control exists or not, one needs to find out if there is any measurable selection for the multifunction such that the minimum value of the cost functional is obtained. Theorems 4.2.1 and 4.3 present the requisite measurability requirements for relaxation theorem application.

4.4 Connections to Stochastic Analysis
The theory developed herein extends naturally to the stochastic setting. A set-valued stochastic process {F (t, omega): t in [0, T], omega in Omega} requires a joint measurability in (t, omega) that is precisely captured by the product sigma-algebra considerations of Theorem 4.4 Stochastic differential inclusions of the form

$$dX(t) \in F(t, X(t))\,dt + G(t, X(t))\,dW(t)$$

where W(t) is a Wiener process, require measurable selections of both F and G. The existence and uniqueness theory for such inclusions.

5. Conclusions
In this study, numerous measurable multifunction theorems have been explored systematically. These include measurable multifunction theorems applicable in Polish spaces, separable Banach spaces, and non-separable Banach spaces. Several fundamental theorems have been discussed in detail. These encompass the Kuratowski-Ryll-Nardzewski Selection Theorem, the Castaing Representation Theorem, generalization of graph measurability to non-separable Banach spaces, as well as Aumann integral and its basic attributes. Applications of these theorems in diverse fields of research have been explored to emphasize their significance in theoretical as well as practical domains. Such applications involve fixed point theory, differential inclusions, optimal control problems, and stochastic differential inclusions.

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